

On the convergence of probabilities of the random graphs' properties expressed by first-order formulae with a bounded quantifier depth¹

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1 Introduction

An asymptotic behavior of the probabilities of Erdős–Rényi random graph first-order properties is studied in the article. In this section we briefly describe the history of the problem and introduce necessary definitions. At the end of the section we formulate our main result.

Let $N \in \mathbb{N}$, $0 \leq p \leq 1$. Denote the set of all undirected graphs without loops and multiple edges with a set of vertices $V_N = \{1, \dots, N\}$ by $\Omega_N = \{G = (V_N, E)\}$. *The Erdős–Rényi random graph* (see [1]–[4]) is a random element $G(N, p)$ of Ω_N with a distribution $P_{N,p}$ on $\mathcal{F}_N = 2^{\Omega_N}$ defined as follows:

$$P_{N,p}(G) = p^{|E|}(1-p)^{C_N^2 - |E|}.$$

The random graph obeys zero-one law with a class of properties \mathcal{C} if for any property $C \in \mathcal{C}$ either $\lim_{N \rightarrow \infty} P_{N,p}(C) = 0$ or $\lim_{N \rightarrow \infty} P_{N,p}(C) = 1$.

The class of first-order properties is the most studied class in this area. Such properties are expressed by first-order formulae (see [5], [6]). These formulae are built of predicate symbols $\sim, =$, logical connectivities $\neg, \Rightarrow, \Leftrightarrow, \vee, \wedge$, variables $x, y, x_1 \dots$ and quantifiers \forall, \exists . Symbols $x, y, x_1 \dots$ express vertices of a graph. The relation symbol \sim expresses the property of two vertices to be adjacent. The symbol $=$ expresses the property of two vertices being coincident. We denote by \mathcal{P} a class of functions $p = p(N)$ such that the random graph $G(N, p)$ obeys zero-one law with the class \mathcal{L} of all first-order properties. In 1969 by Y.V. Glebskii, D.I. Kogan, M.I. Liagonkii and V.A. Talanov in [7] (and independently in 1976 R.Fagin in [8]) proved that if

$$\forall \alpha > 0 \quad N^\alpha \min\{p, 1-p\} \rightarrow \infty, \quad N \rightarrow \infty,$$

then $p \in \mathcal{P}$. Moreover in 1988 S. Shelah and J.H. Spencer (see [9]) expanded the class of functions $p(N)$ “that follow the zero-one law”. They proved that the functions

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$p = N^{-\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha \in (0, 1)$, are in \mathcal{P} . Surely $p = 1 - N^{-\alpha} \in \mathcal{P}$ when $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha \in (0, 1)$.

If α is rational, $0 < \alpha \leq 1$ and $p = N^{-\alpha}$ then $G(N, p)$ does not obey zero-one law (see [4]).

Denote by \mathcal{L}^∞ , $\mathcal{L}^\infty \supset \mathcal{L}$, a class of all properties expressed by formulae containing infinite number of conjunctions and disjunctions. A class \mathcal{L}_k^∞ , $\mathcal{L}_k^\infty \subset \mathcal{L}^\infty$, containing all properties expressed by formulae with quantifier depths bounded by the number k , in the frame of zero-one laws was considered by M. McArthur in 1997 (see [11]). M. McArthur obtained zero-one laws with the class \mathcal{L}_k^∞ for the random graph $G(N, N^{-\alpha})$ with some rational α from $(0, 1]$.

Finally, the random graph $G(N, p)$ does not obey zero-one law with the class \mathcal{L} if $p = N^{-\alpha}$ and α is rational, $\alpha \in (0, 1]$. At the same time the random graph $G(N, p)$ obeys zero-one law with the class \mathcal{L}_k^∞ for some rational $\alpha \in (0, 1]$. Therefore it seems natural to consider the class $\mathcal{L}_k = \mathcal{L} \cap \mathcal{L}_k^\infty$. In 2010 (see [12], [13]) we proved that if $k \geq 3$, $\alpha \in (0, 1/(k-2))$ the random graph $G(N, N^{-\alpha})$ obeys zero-one law with the class \mathcal{L}_k . We also proved that when $\alpha = 1/(k-2)$ the random graph $G(N, N^{-\alpha})$ does not obey zero-one law with this class. This result led us to the following question. Do probabilities $P_{N, N^{-1/(k-2)}}$ of all properties from \mathcal{L}_k converge?

Let us state the main result of the article.

Theorem 1 *Let $k \geq 3$, $p = N^{-\alpha}$, $\alpha = \frac{1}{k-2}$. For any property $L \in \mathcal{L}_k$ there exists $\lim_{N \rightarrow \infty} P_{N, p}(L)$.*

Here we prove Theorem 1 for the case $k \geq 4$ only. The case $k = 3$ is much easier and its correctness can be proved by using the same arguments as in Lemma 1 (see Section 5 and Subsection 8.4).

We give a proof of Theorem 1 in Section 7. This proof is based on a number of statements from Section 5 and Section 6. The mentioned statements are proved in Section 8. The main statement is Lemma 1 which is related to the Ehrenfeucht game (see Section 4). It plays a key role in proofs of zero-one laws and other statements on first-order properties. We introduce all necessary constructions in Section 3 which is divided into 4 subsections. We describe its structure in the end of Section 2 which is devoted to some important and well-known theorems on extensions of small subgraphs in the random graph.

2 Distribution of small subgraphs

For an arbitrary graph G denote by $v(G)$ and $e(G)$ the number of its vertices and the number of its edges respectively. The number $\rho(G) = \frac{e(G)}{v(G)}$ is called *density of G* . The graph G is called *balanced* if for any subgraph $H \subseteq G$ the inequality $\rho(H) \leq \rho(G)$ holds. The graph G is *strictly balanced* if for any subgraph $H \subset G$ the inequality $\rho(H) < \rho(G)$ holds.

Let us describe a problem studied by J.H. Spencer in 1990 (see [4], [14]). Consider graphs $H, G, \tilde{H}, \tilde{G}$. Let $V(H) = \{x_1, \dots, x_k\}$, $V(G) = \{x_1, \dots, x_l\}$, $V(\tilde{H}) = \{\tilde{x}_1, \dots, \tilde{x}_k\}$, $V(\tilde{G}) = \{\tilde{x}_1, \dots, \tilde{x}_l\}$, $H \subset G$, $\tilde{H} \subset \tilde{G}$ (therefore, $k < l$). The graph \tilde{G} is called a (G, H) -extension of the graph \tilde{H} if

$$\{x_{i_1}, x_{i_2}\} \in E(G) \setminus E(H) \Rightarrow \{\tilde{x}_{i_1}, \tilde{x}_{i_2}\} \in E(\tilde{G}) \setminus E(\tilde{H}).$$

If

$$\{x_{i_1}, x_{i_2}\} \in E(G) \setminus E(H) \Leftrightarrow \{\tilde{x}_{i_1}, \tilde{x}_{i_2}\} \in E(\tilde{G}) \setminus E(\tilde{H})$$

then we call \tilde{G} a *strict extension*. Set

$$v(G, H) = |V(G) \setminus V(H)|, \quad e(G, H) = |E(G) \setminus E(H)|,$$

$$f_\alpha(G, H) = v(G, H) - \alpha e(G, H).$$

Fix an arbitrary $\alpha > 0$. If the inequality $f(S, H) > 0$ holds for any graph S such that $H \subset S \subseteq G$ then the pair (G, H) is called α -safe (see [2], [4], [14]). If the inequality $f(G, S) < 0$ holds for any graph S such that $H \subseteq S \subset G$ then the pair (G, H) is called α -rigid (see [2], [4]). The pair (G, H) is called α -neutral if the following three properties hold. For any vertex x of the graph H there exists a vertex of $V(G) \setminus V(H)$ adjacent to x ; $f_\alpha(S, H) > 0$ for any graph S such that $H \subset S \subset G$; $f_\alpha(G, H) = 0$.

Introduce a definition of a maximal pair. Let $\tilde{H} \subset \tilde{G} \subset \Gamma$, $T \subset K$, $|V(T)| \leq |V(\tilde{G})|$. The pair (\tilde{G}, \tilde{H}) is called (K, T) -maximal in Γ if for any subgraph \tilde{T} of \tilde{G} such that $|V(\tilde{T})| = |V(T)|$ and $\tilde{T} \cap \tilde{H} \neq \tilde{T}$ the following property holds. There is no (K, T) -extension \tilde{K} of \tilde{T} in $\Gamma \setminus (\tilde{G} \setminus \tilde{T})$ such that each vertex of $V(\tilde{K}) \setminus V(\tilde{T})$ is not adjacent to any vertex of $V(\tilde{G}) \setminus V(\tilde{T})$.

Let $\alpha \in (0, 1]$. Let a pair (G, H) be α -safe. Let $V(H) = \{x_1, \dots, x_k\}$, $V(G) = \{x_1, \dots, x_l\}$. Denote a set of all α -rigid pairs (K_i, T_i) such that $|V(T_i)| \leq |V(G)|$, $|V(K_i) \setminus V(T_i)| \leq r$ by $\Sigma^{\text{rigid}}(r)$. Consider a set $\Sigma^{\text{neutral}}(r)$ of all α -neutral pairs (K_i, T_i) such that $|V(T_i)| \leq |V(G)|$, $|V(K_i) \setminus V(T_i)| \leq r$.

Consider the random graph $G(N, p)$. Let $H \subset G$, $V(H) = \{x_1, \dots, x_k\}$, $V(G) = \{x_1, \dots, x_l\}$, $\tilde{x}_1, \dots, \tilde{x}_k \in V_N$. Define a random variable $N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k)$ on the probability space $(\Omega_N, \mathcal{F}_N, \mathbf{P}_{N,p})$ as follows. The random variable assigns a number of all (G, H) -extensions induced on the set $\{\tilde{x}_1, \dots, \tilde{x}_k\}$ in \mathcal{G} to a graph \mathcal{G} from Ω_N . A graph X is called a *subgraph of a graph Y induced on a set $S \subset V(Y)$* if $V(X) = S$ and for any vertices $x, y \in S$ the property $\{x, y\} \in E(X) \Leftrightarrow \{x, y\} \in E(Y)$ holds. Let us give a formal definition of $N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k)$. Let $W \subset V_N \setminus \{\tilde{x}_1, \dots, \tilde{x}_k\}$, $|W| = l - k$. If there is a numeration of elements of the set W by numbers $k+1, k+2, \dots, l$ such that the graph $\mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_l\}}$ is a (G, H) -extension of a graph $\mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_k\}}$ then we set $I_W(\mathcal{G}) = 1$. Otherwise we set $I_W(\mathcal{G}) = 0$. The random variable $N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k)$ is defined by the equality

$$N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k) = \sum_{W \subset V_N \setminus \{\tilde{x}_1, \dots, \tilde{x}_k\}, |W|=l-k} I_W.$$

Theorem 2 ([14]) *Let $p = N^{-\alpha}$. Let a pair (G, H) be α -safe. Then*

$$\lim_{N \rightarrow \infty} \mathbf{P}_{N,p}(\forall \tilde{x}_1, \dots, \tilde{x}_k \quad |N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k) - \mathbf{E}_{N,p} N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k)| \leq \varepsilon \mathbf{E}_{N,p} N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k)) = 1$$

for any $\varepsilon > 0$. Here $\mathbf{E}_{N,p}$ is the expectation. Moreover, $\mathbf{E}_{N,p} N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k) = \Theta(N^{f(G,H)})$.

In fact, the statement of this theorem means that almost surely for any vertices $\tilde{x}_1, \dots, \tilde{x}_k$ the relation

$$N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k) \sim \mathbf{E}_{N,p} N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k)$$

holds. In such cases we will use this notation.

In addition to Theorem 2 J.H. Spencer and S. Shelah (see [4], [9]) proved a result on a number of maximal extensions of subgraphs in random graphs (in the case of “prohibited” rigid pairs). In 2010 we extended this result by considering “prohibited” neutral pairs (see [15]).

Let us define new random variables and formulate the corresponding results. Consider a random variable $\hat{N}_{(G,H),r}^{\text{rigid}}(\tilde{x}_1, \dots, \tilde{x}_k)$ such that if $\mathcal{G} \in \Omega_N$ then $\hat{N}_{(G,H),r}^{\text{rigid}}(\tilde{x}_1, \dots, \tilde{x}_k)[\mathcal{G}]$ is the number of strict (G, H) -extensions \tilde{G} of the graph $\tilde{H} = \mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_k\}}$ with the following property. For each pair $(K_i, T_i) \in \Sigma^{\text{rigid}}(r)$ the pair (\tilde{G}, \tilde{H}) is (K_i, T_i) -maximal in \mathcal{G} . First we formulate a result proved by J.H. Spencer and S. Shelah in [9].

Theorem 3 ([9]) *Almost surely for any vertices $\tilde{x}_1, \dots, \tilde{x}_k$*

$$\hat{N}_{(G,H),r}^{rigid}(\tilde{x}_1, \dots, \tilde{x}_k) \sim N_{(G,H)}(\tilde{x}_1, \dots, \tilde{x}_k) \sim \mathbb{E}_{N,p} \hat{N}_{(G,H),r}^{rigid}(\tilde{x}_1, \dots, \tilde{x}_k) = \Theta(N^{f(G,H)}).$$

Recall a result from [15]. Consider a random variable $\hat{N}_{(G,H),r}^{neutral}(\tilde{x}_1, \dots, \tilde{x}_k)$ such that if $\mathcal{G} \in \Omega_N$ then $\hat{N}_{(G,H),r}^{neutral}(\tilde{x}_1, \dots, \tilde{x}_k)[\mathcal{G}]$ is the number of strict (G, H) -extensions \tilde{G} of the graph $\tilde{H} = \mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_k\}}$ with the following property. The pair (\tilde{G}, \tilde{H}) is (K_i, T_i) -maximal in \mathcal{G} for any $(K_i, T_i) \in \Sigma^{neutral}(r)$.

Theorem 4 ([15]) *Almost surely for any vertices $\tilde{x}_1, \dots, \tilde{x}_k$*

$$\hat{N}_{(G,H),r}^{neutral}(\tilde{x}_1, \dots, \tilde{x}_k) \sim \mathbb{E}_{N,p} \hat{N}_{(G,H),r}^{neutral}(\tilde{x}_1, \dots, \tilde{x}_k) = \Theta(N^{f(G,H)}).$$

Let us proceed on to the proof of Theorem 1. An idea of the proof is in the analysis of the probability of the existence of a winning strategy for the second player called Duplicator in the Ehrenfeucht game (see Section 4). In Section 3, all the constructions which are necessary for the proof will be presented.

The next section consists of 4 subsections. The main constructions used in the proof of Lemma 1 are introduced in Subsections 3.3, 3.4. These constructions are maximal for all α -neutral and α -rigid pairs that the first player called Spoiler can build during k rounds. In Subsection 3.2 the notion of a closure $[A]_{\hat{G}}$ for a subgraph A of some graph \hat{G} is introduced.

This closure “contains” all α -neutral pairs. So, in Subsections 3.3, 3.4 graphs containing a maximal number of α -rigid pairs are constructed. Then closures of such graphs are considered. In Subsection 3.1 all necessary pairs of graphs are defined.

3 Constructions

Let $k \geq 4$ be natural. In what follows we assume $\alpha = 1/(k-2)$. So, we will write $f(G, H)$ instead of $f_\alpha(G, H)$ everywhere below.

3.1 Additional graphs

Consider graphs $H_1, H_2, G_1, G_2, G_3^{i_1, \dots, i_t}, G_4, G_4^1, G_4^2$, where $t \in \{1, \dots, k-2\}$, $i_1 \in \{1, \dots, k-2\}$, $i_2 \in \{1, \dots, k-2\} \setminus \{i_1\}$, ..., $i_t \in \{1, \dots, k-2\} \setminus \{i_1, \dots, i_{t-1}\}$.

In the following subsections we will use pairs of these graphs. In fact, we are interested in the pairs (G_1, H_1) , (G_2, H_2) , $(G_3^{i_1, \dots, i_t}, H_2)$, (G_4^1, H_1) , (G_4^2, H_1) .

- 1) The graph G_1 is complete, $V(G_1) = \{x_1, \dots, x_k\}$, H_1 is an arbitrary graph on the set of vertices $V(H_1) = \{x_1, \dots, x_{k-3}\}$.

- 2) Let

$$V(H_2) = \{x_1, \dots, x_{k-3}, x_{k-2}\}, \quad E(H_2) - \text{arbitrary};$$

$$V(G_2) = V(H_2) \cup \{x_{k-1}\}, \quad E(G_2) = E(H_2) \cup \{\{x_1, x_{k-1}\}, \dots, \{x_{k-2}, x_{k-1}\}\}.$$

- 3) Let $t \in \{1, \dots, k-2\}$, $i_1 \in \{1, \dots, k-2\}$, $i_2 \in \{1, \dots, k-2\} \setminus \{i_1\}$, ..., $i_t \in \{1, \dots, k-2\} \setminus \{i_1, \dots, i_{t-1}\}$. Consider graphs $G_3^{i_1, \dots, i_t}$ defined by induction:

$$V(G_3^{i_1}) = V(G_2) \cup \{x_k^{i_1}\},$$

$$E(G_3^{i_1}) = E(G_2) \cup \{\{x_1, x_k^{i_1}\}, \dots, \{x_{k-1}, x_k^{i_1}\}\} \setminus \{\{x_{i_1}, x_k^{i_1}\}\};$$

$$V(G_3^{i_1, \dots, i_t}) = V(G_3^{i_1, \dots, i_{t-1}}) \cup \{x_k^{i_1, \dots, i_t}\},$$

$$E(G_3^{i_1, \dots, i_t}) = E(G_3^{i_1, \dots, i_{t-1}}) \cup \{\{x_1, x_k^{i_1, \dots, i_t}\}, \dots, \{x_{k-1}, x_k^{i_1, \dots, i_t}\}\} \setminus \{\{x_{i_t}, x_k^{i_1, \dots, i_t}\}\}.$$

- 4) Let

$$V(G_4) = V(H_1) \cup \{x_{k+1}, x_{k+2}, x_{k+3}\},$$

$$V(G_4^1) = V(G_4) \cup \{x_{k+4}^1\}, \quad V(G_4^2) = V(G_4) \cup \{x_{k+4}^2, x_{k+5}^2\};$$

$$E(G_4) = E(H_1) \cup \{\{x_1, x_{k+1}\}, \dots, \{x_{k-4}, x_{k+1}\}, \{x_1, x_{k+2}\}, \dots, \{x_{k-3}, x_{k+2}\},$$

$$\{x_1, x_{k+3}\}, \dots, \{x_{k-3}, x_{k+3}\}, \{x_{k+1}, x_{k+2}\}, \{x_{k+1}, x_{k+3}\}, \{x_{k+2}, x_{k+3}\}\};$$

$$E(G_4^1) = E(G_4) \cup \{\{x_1, x_{k+4}^1\}, \dots, \{x_{k-3}, x_{k+4}^1\}, \{x_{k+1}, x_{k+4}^1\}, \{x_{k+3}, x_{k+4}^1\}\};$$

$$E(G_4^2) = E(G_4) \cup \{\{x_1, x_{k+4}^2\}, \dots, \{x_{k-3}, x_{k+4}^2\}, \{x_1, x_{k+5}^2\}, \dots, \{x_{k-3}, x_{k+5}^2\},$$

$$\{x_{k+1}, x_{k+4}^2\}, \{x_{k+1}, x_{k+5}^2\}, \{x_{k+4}^2, x_{k+5}^2\}\}.$$

Let $t \in \{1, \dots, k-2\}$, $i_1 \in \{1, \dots, k-2\}$, $i_2 \in \{1, \dots, k-2\} \setminus \{i_1\}$, ..., $i_t \in \{1, \dots, k-2\} \setminus \{i_1, \dots, i_{t-1}\}$. Consider the set S^{i_1, \dots, i_t} of all unordered collections of $k-2$ vertices from $V(G_3^{i_1, \dots, i_t})$. For each $U \subset S^{i_1, \dots, i_t}$ consider the union of the graph $G_3^{i_1, \dots, i_t}$ and all the (G_2, H_2) -extensions of graphs $G_3^{i_1, \dots, i_t}|_{\mathbf{u}}$, $\mathbf{u} \in U$. Denote this union by $G_3^{i_1, \dots, i_t}(U)$. Note that a union of a graph on $k-2$ vertices with its (G_2, H_2) -extension is obtained by adding one vertex adjacent to all its vertices. Let $\mathcal{U}_i^{i_1, \dots, i_t}$ be a set of all subsets of S^{i_1, \dots, i_t} with the cardinality i .

Let us construct the closure of a graph.

3.2 The closure $[A]_{\widehat{G}}$ in \widehat{G} of a graph A

Consider arbitrary vertices $\widehat{x}_1, \dots, \widehat{x}_{k-3}$. Let \widehat{G} be a graph. Let $\widehat{x}_1, \dots, \widehat{x}_{k-3}$ be the vertices of the graph \widehat{G} . Consider any graph $A \subset \widehat{G}$ on a set of vertices $V(A) = \{a_1, \dots, a_d\}$, $d \geq k-2$.

First of all let us note that for the graph A there exist several closures in the graph \widehat{G} . All these closures are isomorphic.

We construct the graph $[A]_{\widehat{G}}$ in $k-1$ steps. Let S be a set of all different unordered collections of $k-2$ vertices from $V(A)$. Set $[A]_{\widehat{G}} = A$, $\widehat{G}_1 = \widehat{G}$.

The first step is divided into $|S^{1, \dots, k-2}|$ parts. Consider the first part of the step. Let $\{a_{i_1}, \dots, a_{i_{k-2}}\} \in S$, $U \in \mathcal{U}_{|S^{1, \dots, k-2}|}^{1, \dots, k-2} = \{S^{1, \dots, k-2}\}$. Assume that there exists an $(G_3^{1, \dots, k-2}(U), H_2)$ -extension \widehat{Q} of $A|_{\{a_{i_1}, \dots, a_{i_{k-2}}\}}$ in \widehat{G}_1 . We add only one such extension to the graph $[A]_{\widehat{G}}$ and for all these extensions we remove extenders from the graph \widehat{G}_1 (if a graph X is an extension of a graph Y then we say that graph $X \setminus Y$ is *an extender*).

Let the first s parts, $s \leq |S^{1, \dots, k-2}| - 1$, of the first step of the graphs $[A]_{\widehat{G}}, \widehat{G}_1$ construction be done. Let us describe the $s+1$ -th part. Let $\{a_{i_1}, \dots, a_{i_{k-2}}\} \in S$, $U \in \mathcal{U}_{|S^{1, \dots, k-2}| - s}^{1, \dots, k-2}$. Assume that an $(G_3^{1, \dots, k-2}(U), H_2)$ -extension \widehat{Q} of $A|_{\{a_{i_1}, \dots, a_{i_{k-2}}\}}$ in \widehat{G}_1 exists. We add only one such extension to the graph $[A]_{\widehat{G}}$ and for all these extensions we remove the extenders from the graph \widehat{G}_1 .

Let the i -th step, $i \leq k-3$, be done. Describe **the $i+1$ -th step**. We divide this step into $|S^{1, \dots, k-2-i}|$ parts.

Let $\{a_{i_1}, \dots, a_{i_{k-2-i}}\} \in S$, $i_1, \dots, i_{k-2-i} \in \{1, \dots, k-2\}$ be an ordered collection of different numbers, U be a subset of $\mathcal{U}_{|S^{i_1, \dots, i_{k-2-i}}|}^{i_1, \dots, i_{k-2-i}}$. Let an $(G_3^{i_1, \dots, i_{k-2-i}}(U), H_2)$ -extension \widehat{Q} of $A|_{\{a_{i_1}, \dots, a_{i_{k-2-i}}\}}$ in \widehat{G}_1 exist. We add only one such extension to the graph $[A]_{\widehat{G}}$ and for all these extensions we remove the extenders from the graph \widehat{G}_1 .

Let the first s parts, $s \leq |S^{i_1, \dots, i_{k-2-i}}| - 1$, of the $i+1$ -th step be done, $\{a_{i_1}, \dots, a_{i_{k-2-i}}\} \in S$. Let $i_1, \dots, i_{k-2-i} \in \{1, \dots, k-2\}$ be an unordered collection of different numbers. Let $U \in \mathcal{U}_{|S^{i_1, \dots, i_{k-2-i}}| - s}^{i_1, \dots, i_{k-2-i}}$. Assume that an $(G_3^{i_1, \dots, i_{k-2-i}}(U), H_2)$ -extension \widehat{Q} of $A|_{\{a_{i_1}, \dots, a_{i_{k-2-i}}\}}$ in \widehat{G}_1 exists. We add only one such extension to the graph $[A]_{\widehat{G}}$ and for all these extensions we remove the extenders from the graph \widehat{G}_1 .

Describe the final $k-1$ -th step. Let $\{a_{i_1}, \dots, a_{i_{k-2}}\} \in S$. Assume that there exists a (G_2, H_2) -extension \widehat{Q} of $A|_{\{a_{i_1}, \dots, a_{i_{k-2}}\}}$ in \widehat{G}_1 . We add only one such extension to the graph $[A]_{\widehat{G}}$. The graph $[A]_{\widehat{G}}$ is constructed.

In the following two subsections we construct graphs $X_{\widehat{G}}^l(\widehat{x}_1, \dots, \widehat{x}_{k-3})$, $\widehat{X}_{\widehat{G}}^l(\widehat{x}_1, \dots, \widehat{x}_{k-3})$, $X_j^l(\widehat{x}_1)$, $\widehat{X}_j^l(\widehat{x}_1)$, where $l \in \{1, 2, 3, 4, 5\}$, j is from some set J . The graphs $\widehat{X}_j^l(\widehat{x}_1)$ are subgraphs of some graphs \widehat{G}_{i_j} which are chosen in such a way that these subgraphs are “different” in some sense. The graphs $\widehat{X}_{\widehat{G}}^l(\widehat{x}_1, \dots, \widehat{x}_{k-3})$ are subgraphs of the graph \widehat{G} and built as l grows from 1 to 5. The graph $\widehat{X}_j^l(\widehat{x}_1)$ is the union of the graphs $\widehat{X}_{\widehat{G}_{i_j}}^l(\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_{k-3})$ over some sets of vertices $\widehat{x}_2, \dots, \widehat{x}_{k-3}$ of the graph \widehat{G}_{i_j} . Finally, graphs $X_{\widehat{G}}^l(\widehat{x}_1, \dots, \widehat{x}_{k-3})$, $X_j^l(\widehat{x}_1)$ are the unions of the closures of some subgraphs of $\widehat{X}_{\widehat{G}}^l(\widehat{x}_1, \dots, \widehat{x}_{k-3})$, $\widehat{X}_j^l(\widehat{x}_1)$ respectively.

3.3 Graphs $X_{\widehat{G}}^l(\widehat{x}_1, \dots, \widehat{x}_{k-3})$, $\widehat{X}_{\widehat{G}}^l(\widehat{x}_1, \dots, \widehat{x}_{k-3})$, $l \in \{1, 2, 3, 4, 5\}$

Assume that the graph \widehat{G} considered in the previous subsection does not contain subgraphs W with $v(W) < k^3$ and $\rho(W) > k-2$. We consider this restriction because of the following reasonings. As it is mentioned in Section 2 the constructions we build should contain a maximal number of rigid pairs. Without a restriction on the density of subgraphs in \widehat{G} the number of such pairs in \widehat{G} can be arbitrarily large. We choose the number $k-2$ as in the random graph $G(N, p)$ there are no subgraphs W with $v(W)$ bounded by a fixed number and $\rho(W) > k-2$.

We construct the subgraph $\hat{X}_G^1(\hat{x}_1, \dots, \hat{x}_{k-3})$ of the graph \hat{G} by adding to the graph $\hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}$ its (G_1, H_1) -extensions by the following rule. Consider all pairs of (G_1, H_1) -extensions (A, B) of $\hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}$ in \hat{G} such that $(E(A) \setminus E(\hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}})) \cap (E(B) \setminus E(\hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}})) \neq \emptyset$. If such pairs exist (we say that extensions from such pairs are *intersecting*) we take their union and denote it by $\hat{X}_G^1(\hat{x}_1, \dots, \hat{x}_{k-3})$.

Suppose that the number of all the (G_1, H_1) -extensions such that there exist other (G_1, H_1) -extensions which intersect them is greater than $2(k-3)(k-2)$. Let us prove that $\rho(\hat{X}_G^1(\hat{x}_1, \dots, \hat{x}_{k-3})) > k-2$.

Let us reconsider the construction of the union $\hat{X}_G^1(\hat{x}_1, \dots, \hat{x}_{k-3})$. Here we assume that the extensions are added step by step. At each step we add either an extension intersecting extensions added earlier or an intersecting pair of extensions which does not intersect extensions added earlier. Let v_i be a number of vertices added at the i -th step. Let $h > (k-3)(k-2)$ be a number of steps. Then

$$\frac{(k-2)(v_1 + \dots + v_h) + h}{v_1 + \dots + v_h + k-3} > k-2.$$

Therefore, $\rho(\hat{X}_G^1(\hat{x}_1, \dots, \hat{x}_{k-3})) > k-2$. Thus, the number of intersecting (G_1, H_1) -extensions is not greater than $2(k-3)(k-2)$.

If in the graph \hat{G} there is no intersecting (G_1, H_1) -extensions of $\hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}$ we set

$$\hat{X}_G^1(\hat{x}_1, \dots, \hat{x}_{k-3}) = \hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}.$$

Let

$$X_G^1(\hat{x}_1, \dots, \hat{x}_{k-3}) = [\hat{X}_G^1(\hat{x}_1, \dots, \hat{x}_{k-3})]_{\hat{G}}.$$

Consider a set \mathcal{M} of all pairs $([M]_{\hat{G}}, \hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}})$, where M is an (G_1, H_1) -extension of the graph $\hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}$ in \hat{G} without vertices of the graph $X_G^1(\hat{x}_1, \dots, \hat{x}_{k-3}) \setminus \hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}$. Choose from the set \mathcal{M} a collection of non-isomorphic pairs

$$([M_1]_{\hat{G}}, \hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}), \dots, ([M_\tau]_{\hat{G}}, \hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}),$$

such that τ is maximal (we call some pairs of graphs $(A_1, B_1), (A_2, B_2)$, with $V(A_1) = \{a_1^1, \dots, a_n^1\}$, $V(A_2) = \{a_1^2, \dots, a_n^2\}$, $V(B_1) = \{a_1^1, \dots, a_m^1\}$, $V(B_2) = \{a_1^2, \dots, a_m^2\}$, $m < n$, *isomorphic* if $\{a_i^1, a_j^1\} \in E(A_1) \setminus E(B_1) \Leftrightarrow \{a_i^2, a_j^2\} \in E(A_2) \setminus E(B_2)$). Let

$$X_G^2(\hat{x}_1, \dots, \hat{x}_{k-3}) = [M_1]_{\hat{G}} \cup \dots \cup [M_\tau]_{\hat{G}} \cup X_G^1(\hat{x}_1, \dots, \hat{x}_{k-3}),$$

$$\hat{X}_G^2(\hat{x}_1, \dots, \hat{x}_{k-3}) = M_1 \cup \dots \cup M_\tau \cup \hat{X}_G^1(\hat{x}_1, \dots, \hat{x}_{k-3}).$$

Similarly to the case of intersecting (G_1, H_1) -extensions, the number of extenders of strict (G_4^1, H_1) - and (G_4^2, H_1) -extensions of $\widehat{G}|_{\{\widehat{x}_1, \dots, \widehat{x}_{k-3}\}}$ in \widehat{G} which have common vertices with other such extenders is less than or equal to $2(k-3)(k-2)$. Let W_1 be the union of all such extensions. Set $W_2 = [W_1]_{\widehat{G}}$,

$$X_{\widehat{G}}^3(\widehat{x}_1, \dots, \widehat{x}_{k-3}) = W_2 \cup X_{\widehat{G}}^2(\widehat{x}_1, \dots, \widehat{x}_{k-3}), \quad \widehat{X}_{\widehat{G}}^3(\widehat{x}_1, \dots, \widehat{x}_{k-3}) = W_1 \cup \widehat{X}_{\widehat{G}}^2(\widehat{x}_1, \dots, \widehat{x}_{k-3}).$$

Consider a set \mathcal{M} of all pairs $([M]_{\widehat{G}}, \widehat{G}|_{\{\widehat{x}_1, \dots, \widehat{x}_{k-3}\}})$, where M is (G_4^1, H_1) - or (G_4^2, H_1) -extension of the graph $\widehat{G}|_{\{\widehat{x}_1, \dots, \widehat{x}_{k-3}\}}$ in $\widehat{G} \setminus (W_2 \setminus \widehat{G}|_{\{\widehat{x}_1, \dots, \widehat{x}_{k-3}\}})$ having no vertices from $V(X_{\widehat{G}}^3(\widehat{x}_1, \dots, \widehat{x}_{k-3}) \setminus \widehat{G}|_{\{\widehat{x}_1, \dots, \widehat{x}_{k-3}\}})$. Choose non-isomorphic pairs

$$([M_1]_{\widehat{G}}, \widehat{G}|_{\{\widehat{x}_1, \dots, \widehat{x}_{k-3}\}}), \dots, ([M_\tau]_{\widehat{G}}, \widehat{G}|_{\{\widehat{x}_1, \dots, \widehat{x}_{k-3}\}})$$

from the set \mathcal{M} in such a way that the number τ is maximal. Set

$$X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}) = X_{\widehat{G}}^3(\widehat{x}_1, \dots, \widehat{x}_{k-3}) \cup [M_1]_{\widehat{G}} \cup \dots \cup [M_\tau]_{\widehat{G}},$$

$$\widehat{X}_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}) = \widehat{X}_{\widehat{G}}^3(\widehat{x}_1, \dots, \widehat{x}_{k-3}) \cup M_1 \cup \dots \cup M_\tau.$$

For each vertex $\widehat{x} \in V(\widehat{G}) \setminus V(X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}))$ adjacent to less than or equal to $k-5$ vertices of $\widehat{x}_1, \dots, \widehat{x}_{k-3}$ consider a set $\Upsilon(\widehat{x})$ containing \widehat{x} and all vertices $\widehat{x}^1 \in V(\widehat{G}) \setminus V(X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}))$, satisfying the following property. There exists a vertex $\widehat{x}^2 \in V(\widehat{G}) \setminus V(X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}))$ such that $\widehat{x}^1 \sim \widehat{x}^2$, $\widehat{x}^1 \sim \widehat{x}$, $\widehat{x}^2 \sim \widehat{x}$, $\widehat{x}^1 \sim \widehat{x}_i$, $\widehat{x}^2 \sim \widehat{x}_i$, $i \in \{1, \dots, k-3\}$. Let Υ be a union of sets $\Upsilon(\widehat{x})$ over all \widehat{x} such that

$$f(X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}) \cup \widehat{G}|_{\Upsilon(\widehat{x})}, X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3})) < 0.$$

Consider also a set \mathcal{M} of all pairs $(X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}) \cup [\widehat{G}|_{\Upsilon(\widehat{x}) \cup \{\widehat{x}_1, \dots, \widehat{x}_{k-3}\}}]_{\widehat{G}}, X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}))$, where

$$f(X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}) \cup \widehat{G}|_{\Upsilon(\widehat{x})}, X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3})) = 0.$$

Let $([M_1]_{\widehat{G}}, X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3})), \dots, ([M_\tau]_{\widehat{G}}, X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}))$ be non-isomorphic pairs from \mathcal{M} such that the number τ is maximal. Set

$$X_{\widehat{G}}^5(\widehat{x}_1, \dots, \widehat{x}_{k-3}) = [M_1]_{\widehat{G}} \cup \dots \cup [M_\tau]_{\widehat{G}} \cup X_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}) \cup [\widehat{G}|_{\Upsilon \cup \{\widehat{x}_1, \dots, \widehat{x}_{k-3}\}}]_{\widehat{G}},$$

$$\widehat{X}_{\widehat{G}}^5(\widehat{x}_1, \dots, \widehat{x}_{k-3}) = M_1 \cup \dots \cup M_\tau \cup \widehat{X}_{\widehat{G}}^4(\widehat{x}_1, \dots, \widehat{x}_{k-3}) \cup \widehat{G}|_{\Upsilon} \widehat{X}_{\widehat{G}}^5(\widehat{x}_1, \dots, \widehat{x}_{k-3}).$$

3.4 Graphs $X_j^l(\hat{x}_1), \hat{X}_j^l(\hat{x}_1), l \in \{1, 2, 3, 4, 5\}$

Let $\hat{G}_1, \hat{G}_2, \dots$ be graphs satisfying the following properties:

- $\bigcap_{i=1}^{\infty} V(\hat{G}_i) \supseteq \{\hat{x}_1, \dots, \hat{x}_{k-3}\}$;
- pairs $(\hat{G}_1, \hat{G}_1|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}), (\hat{G}_2, \hat{G}_2|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}), \dots$ are pairwise non-isomorphic;
- $\rho(X_{\hat{G}_i}^5(\hat{x}_1, \dots, \hat{x}_{k-3})) \leq k - 2$ for every $i \in \mathbb{N}$;
- there is no such a graph \hat{G}_0 that the graphs $\hat{G}_0, \hat{G}_1, \hat{G}_2, \dots$ satisfy the first three properties.

Then obviously there exists a final set $\{i_1, \dots, i_{a(k)}\}$ such that pairs

$$(X_{\hat{G}_{i_j}}^5(\hat{x}_1, \dots, \hat{x}_{k-3}), \hat{G}_{i_j}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}}), \quad j \in \{1, \dots, a(k)\}$$

are pairwise non-isomorphic. Moreover, for any $i \in \mathbb{N}$ there exists $j \in \{1, \dots, a(k)\}$ such that the pairs $(X_{\hat{G}_i}^5(\hat{x}_1, \dots, \hat{x}_{k-3}), \hat{G}_i|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}})$ and $(X_{\hat{G}_{i_j}}^5(\hat{x}_1, \dots, \hat{x}_{k-3}), \hat{G}_{i_j}|_{\{\hat{x}_1, \dots, \hat{x}_{k-3}\}})$ are isomorphic.

For any $j \in \{1, \dots, a(k)\}$ and $l \in \{1, 2, 3, 4, 5\}$ set

$$X_j^l(\hat{x}_1, \dots, \hat{x}_{k-3}) = X_{\hat{G}_{i_j}}^l(\hat{x}_1, \dots, \hat{x}_{k-3}).$$

Consider the graph \hat{G} . Let $\xi \in \{1, \dots, k-4\}$ be fixed. Let Y^1, \dots, Y^t be subgraphs of the subgraph \hat{G} satisfying the following properties.

- For each $i \in \{1, \dots, t\}$ there exist $j \in \{1, \dots, a(k)\}$ and vertices $\hat{x}_{\xi+1}^i, \dots, \hat{x}_{k-3}^i$ of the graph \hat{G} such that $Y^i = X_j^5(\hat{x}_1, \dots, \hat{x}_{\xi}, \hat{x}_{\xi+1}^i, \dots, \hat{x}_{k-3}^i)$.
- Take arbitrary $i_1, i_2 \in \{1, \dots, t\}$, $i_1 \neq i_2$, such that $\hat{x}_{\xi+1}^{i_1}, \hat{x}_{\xi+1}^{i_2}$ do not coincide for some $i \in \{\xi+1, \dots, k-4\}$. Let $\mu \in \{\xi, \dots, k-4\}$ be such that for all $i \in \{\xi+1, \dots, \mu\}$ the vertices $\hat{x}_{\xi+1}^{i_1}, \hat{x}_{\xi+1}^{i_2}$ coincide, and the vertices $\hat{x}_{\mu+1}^{i_1}, \hat{x}_{\mu+1}^{i_2}$ are different. Then $f(\bigcup_{i \in I_1^{i_1}} Y^i, \bigcap_{i \in I_1^{i_1}} Y^i) \leq 0$, where $I_1^{i_1} = \{u : \forall i \in \{\xi+1, \dots, \mu\} \hat{x}_i^u = \hat{x}_i^{i_1}, \hat{x}_{\mu+1}^u \neq \hat{x}_{\mu+1}^{i_1}\}$.
- Pairs $(\bigcup_{i \in I_1^{i_1}} Y^i, \hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{\xi}, \hat{x}_{\xi+1}^{i_1}, \dots, \hat{x}_{\mu}^{i_1}\}}), (\bigcup_{i \in I_2^{i_2}} Y^i, \hat{G}|_{\{\hat{x}_1, \dots, \hat{x}_{\xi}, \hat{x}_{\xi+1}^{i_2}, \dots, \hat{x}_{\mu}^{i_2}\}})$ are non-isomorphic for any $\mu \in \{\xi, \dots, k-4\}$ and any $i_1, i_2 \in \{1, \dots, t\}$ such that $\hat{x}_{\mu+1}^{i_1} \neq \hat{x}_{\mu+1}^{i_2}$. Here $I_1^{i_1} = \{u : \forall i \in \{\xi+1, \dots, \mu+1\} \hat{x}_i^u = \hat{x}_i^{i_1}\}$, $I_2^{i_2} = \{u : \forall i \in \{\xi+1, \dots, \mu\} \hat{x}_i^u = \hat{x}_i^{i_2}, \hat{x}_{\mu+1}^u = \hat{x}_{\mu+1}^{i_2}\}$.

If in the graph \widehat{G} there is no subgraph Y^{t+1} different from Y^1, \dots, Y^t and such that Y^1, \dots, Y^{t+1} satisfy the three properties described above, then we denote $Y^1 \cup \dots \cup Y^t$ by $X_{\widehat{G}}^5(\widehat{x}_1, \dots, \widehat{x}_\xi)$. If $t = 0$ set $X_{\widehat{G}}^5(\widehat{x}_1, \dots, \widehat{x}_\xi) = \widehat{G}|_{\{\widehat{x}_1, \dots, \widehat{x}_\xi\}}$.

If, in addition, graphs Y^1, \dots, Y^t follow the properties described below then we say that the graph $X_{\widehat{G}}^5(\widehat{x}_1, \dots, \widehat{x}_\xi)$ is $(\widehat{x}_1, \dots, \widehat{x}_\xi)$ -net in \widehat{G} .

- For any $i_1, i_2 \in \{1, \dots, t\}$, $i_1 \neq i_2$, the set $V(Y^1) \cap V(Y^2)$ is a subset of $\{\widehat{x}_1, \dots, \widehat{x}_\xi, \widehat{x}_{\xi+1}^{i_1}, \dots, \widehat{x}_{k-4}^{i_1}\} \cap \{\widehat{x}_1, \dots, \widehat{x}_\xi, \widehat{x}_{\xi+1}^{i_2}, \dots, \widehat{x}_{k-4}^{i_2}\}$.
- For any $i_1, i_2 \in \{1, \dots, t\}$, $i_1 \neq i_2$, either the sets $\{\widehat{x}_{\xi+1}^{i_1}, \dots, \widehat{x}_{k-3}^{i_1}\}, \{\widehat{x}_{\xi+1}^{i_2}, \dots, \widehat{x}_{k-3}^{i_2}\}$ do not intersect each other, or there exists $\mu \in \{\xi+1, \dots, k-4\}$ such that for any $i \in \{\xi+1, \dots, \mu\}$ the vertices $\widehat{x}_i^{i_1}, \widehat{x}_i^{i_2}$ coincide and the sets $\{\widehat{x}_{\mu+1}^{i_1}, \dots, \widehat{x}_{k-3}^{i_1}\}, \{\widehat{x}_{\mu+1}^{i_2}, \dots, \widehat{x}_{k-3}^{i_2}\}$ do not intersect each other.
- For any $i_1, i_2 \in \{1, \dots, t\}$, $i_1 \neq i_2$, the set $E(Y_{i_1} \cup Y_{i_2}) \setminus (E(Y_{i_1}) \cup E(Y_{i_2}))$ is empty.

Let us consider the following situation. There exist graphs $\widehat{G}^1, \widehat{G}^2$ and vertices $\widehat{x}_1^1, \dots, \widehat{x}_\xi^1, \widehat{x}_1^2, \dots, \widehat{x}_\xi^2$ such that

- $X_{\widehat{G}^1}^5(\widehat{x}_1^1, \dots, \widehat{x}_\xi^1) = Y_1^1 \cup \dots \cup Y_1^t$, $X_{\widehat{G}^2}^5(\widehat{x}_1^2, \dots, \widehat{x}_\xi^2) = Y_2^1 \cup \dots \cup Y_2^t$;
- for any $i \in \{1, \dots, t\}$ there exists $j \in \{1, \dots, a(k)\}$ such that $Y_1^i = X_j^5(\widehat{x}_1^1, \dots, \widehat{x}_\xi^1, \widehat{x}_{\xi+1}^{i,1}, \dots, \widehat{x}_{k-3}^{i,1})$, $Y_2^i = X_j^5(\widehat{x}_1^2, \dots, \widehat{x}_\xi^2, \widehat{x}_{\xi+1}^{i,2}, \dots, \widehat{x}_{k-3}^{i,2})$ for some vertices $\widehat{x}_{\xi+1}^{i,1}, \dots, \widehat{x}_{k-3}^{i,1}, \widehat{x}_{\xi+1}^{i,2}, \dots, \widehat{x}_{k-3}^{i,2}$;
- for any different $i_1, i_2 \in \{1, \dots, t\}$ there exists $\mu \in \{\xi, \dots, k-4\}$ such that

$$\widehat{x}_{j+1}^{i_1,1} = \widehat{x}_{j+1}^{i_2,1}, \widehat{x}_{j+1}^{i_1,2} = \widehat{x}_{j+1}^{i_2,2}, \quad j \in \{\xi, \dots, \mu-1\},$$

$$\widehat{x}_{\mu+1}^{i_1,1} \neq \widehat{x}_{\mu+1}^{i_2,1}, \widehat{x}_{\mu+1}^{i_1,2} \neq \widehat{x}_{\mu+1}^{i_2,2};$$

- the graph $X_{\widehat{G}^2}^5(\widehat{x}_1^2, \dots, \widehat{x}_\xi^2)$ is an $(\widehat{x}_1^2, \dots, \widehat{x}_\xi^2)$ -net in \widehat{G}^2 .

In this case we say that the graph $X_{\widehat{G}^2}^5(\widehat{x}_1^2, \dots, \widehat{x}_\xi^2)$ is a *net of the graph* $X_{\widehat{G}^1}^5(\widehat{x}_1^1, \dots, \widehat{x}_\xi^1)$. For any $i \in \{1, \dots, t\}$, $\mu \in \{\xi+1, \dots, k-3\}$ we introduce a notation

$$\widehat{x}_\mu^{i,2} = \text{NET}_{\widehat{G}^1, \widehat{G}^2, \widehat{x}_1^1, \dots, \widehat{x}_\xi^1, \widehat{x}_1^2, \dots, \widehat{x}_\xi^2}(\widehat{x}_\mu^{i,1}).$$

Note that the function NET is defined on the set of symbols $\{\widehat{x}_\mu^{i,1}, i \in \{1, \dots, t\}, \mu \in \{\xi+1, \dots, k-3\}\}$ of cardinality $t(k-3-\xi)$. It means that some vertices from the

set of vertices $\{\widehat{x}_\mu^{i,1}, i \in \{1, \dots, t\}, \mu \in \{\xi + 1, \dots, k - 3\}\}$ can be equal, but the corresponding symbols are different. In this cases the function NET assigns a vertex to several different vertices.

Obviously the number of different (\widehat{x}_1) -nets (up to isomorphism) in \widehat{G} for different graphs \widehat{G} with *maximal density* greater than or equal to $k - 2$ is finite (maximal density of a graph G equals $\rho^{\max}(G) := \max_{H \subseteq G} \{\rho(H)\}$). Let $X_1^5(\widehat{x}_1), \dots, X_{\widetilde{m}(k)}^5(\widehat{x}_1)$ be such nets with the following property. For any net $X_0^5(\widehat{x}_i)$ in \widehat{G} with maximal density greater than or equal to $k - 2$ there exists a number $j \in \{1, \dots, \widetilde{m}(k)\}$ such that the mapping $X_0^5(\widehat{x}_i) \rightarrow X_j^5(\widehat{x}_i)$ preserving the vertex order is an isomorphism. Let graphs $X_1^5(\widehat{x}_1), \dots, X_{\widetilde{m}(k)}^5(\widehat{x}_1)$ be ordered so that maximal densities of the graphs $X_1^5(\widehat{x}_1), \dots, X_{\widetilde{m}(k)}^5(\widehat{x}_1)$ are equal to $k - 2$ and the densities of graphs $X_1^5(\widehat{x}_1), \dots, X_{\widetilde{m}(k)}^5(\widehat{x}_1)$ are equal to $k - 2$. For each number $j \in \{1, \dots, \widetilde{m}(k)\}$ let $X_j^5(\widehat{x}_1) = Y_j^1 \cup \dots \cup Y_j^{t(j)}$ be a decomposition defined by properties described above (definition of an (\widehat{x}_1) -net). Let $j \in \{1, \dots, \widetilde{m}(k)\}$, $s \in \{1, \dots, t(j)\}$. Denote by $i^{j,s}$ a number from $\{1, \dots, a(k)\}$ such that $Y_j^s = X_{ij,s}^5(\widehat{x}_1, \widehat{x}_2^{j,s}, \dots, \widehat{x}_{k-3}^{j,s})$ for some vertices $\widehat{x}_2^{j,s}, \dots, \widehat{x}_{k-3}^{j,s}$. For every $l \in \{1, 2, 3, 4\}$ set

$$\begin{aligned} X_j^l(\widehat{x}_1) &= X_{ij,1}^l(\widehat{x}_1, \widehat{x}_2^{j,1}, \dots, \widehat{x}_{k-3}^{j,1}) \cup \dots \cup X_{ij,t(j)}^l(\widehat{x}_1, \widehat{x}_2^{j,t(j)}, \dots, \widehat{x}_{k-3}^{j,t(j)}), \\ \widehat{X}_j^l(\widehat{x}_1) &= \widehat{X}_{ij,1}^l(\widehat{x}_1, \widehat{x}_2^{j,1}, \dots, \widehat{x}_{k-3}^{j,1}) \cup \dots \cup \widehat{X}_{ij,t(j)}^l(\widehat{x}_1, \widehat{x}_2^{j,t(j)}, \dots, \widehat{x}_{k-3}^{j,t(j)}), \\ \widehat{X}_j^5(\widehat{x}_1) &= \widehat{X}_{ij,1}^5(\widehat{x}_1, \widehat{x}_2^{j,1}, \dots, \widehat{x}_{k-3}^{j,1}) \cup \dots \cup \widehat{X}_{ij,t(j)}^5(\widehat{x}_1, \widehat{x}_2^{j,t(j)}, \dots, \widehat{x}_{k-3}^{j,t(j)}). \end{aligned}$$

The definition of the graph $X_G^5(\widehat{x}_1, \dots, \widehat{x}_\xi)$ implies that there exists an analogous decomposition:

$$X_G^5(\widehat{x}_1, \dots, \widehat{x}_\xi) = \bigcup_{i=1}^{t(\widehat{x}_1, \dots, \widehat{x}_\xi, \widehat{G})} X_G^5(\widehat{x}_1, \dots, \widehat{x}_\xi, \widehat{x}_{\xi+1}^i(\widehat{x}_1, \dots, \widehat{x}_\xi), \dots, \widehat{x}_{k-3}^i(\widehat{x}_1, \dots, \widehat{x}_\xi)).$$

Note that any graph G with maximal density ρ has subgraphs H_1, H_2 , $H_1 \subseteq H_2$ such that $\rho(H_1) = \rho(H_2) = \rho$, the graph H_1 is strictly balanced, the pair (H_2, H_1) is $1/\rho$ -neutral chain, and either the pair (G, H_2) is $1/\rho$ -safe or $H_2 = G$ (a pair (H_2, H_1) is called α -neutral chain if $H_2 \supseteq H_1$ and there exist graphs $K_1, \dots, K_r, T_1, \dots, T_{r-1}$ with the following properties: $H_1 = K_1 \subset K_2 \subset \dots \subset K_r = H_2$; $T_i \subset K_i$, $i \in \{1, \dots, r - 1\}$; pairs $((K_i \setminus K_{i-1}) \cup T_{i-1}, T_{i-1})$, $i \in \{2, \dots, r\}$ are α -neutral; for any $i \in \{2, \dots, r\}$ there are no edges connecting vertices of the graph $K_i \setminus K_{i-1}$ and vertices of the graph $K_{i-1} \setminus T_{i-1}$).

Denote by $X_j^*(\hat{x}_1)$, $X_j^{**}(\hat{x}_1)$ the corresponding subgraphs of the graph $X_j^5(\hat{x}_1)$ for each $j \in \{1, \dots, \widehat{m}(k)\}$. The graph $X_j^*(\hat{x}_1)$ is strictly balanced with density $\rho^{\max}(X_j^5(\hat{x}_1))$. The pair $(X_j^{**}(\hat{x}_1), X_j^*(\hat{x}_1))$ is $1/\rho^{\max}(X_j^5(\hat{x}_1))$ -neutral chain. Note that when $j \in \{1, \dots, \widehat{m}(k)\}$ the graphs $X_j^{**}(\hat{x}_1)$, $X_j^5(\hat{x}_1)$ are equal.

Let us prove that the graph $X_j^*(\hat{x}_1)$ contains the vertex \hat{x}_1 if $j \in \{1, \dots, \widehat{m}(k)\}$. It is easy to see that in the graph $X_j^*(\hat{x}_1)$ there are at least $k-1$ vertices such that any vertex \hat{x}^0 of them follows the property described below. There exists a number $l \in \{1, \dots, t(j)\}$ such that the vertices $\hat{x}_1, \hat{x}_2^{j,l}, \dots, \hat{x}_{k-3}^{j,l}$ are adjacent to \hat{x}^0 in the graph $X_j^5(\hat{x}_1)$ and $\hat{x}^0 \in V(X_{i,j,l}^5(\hat{x}_1, \hat{x}_2^{j,l}, \dots, \hat{x}_{k-3}^{j,l}))$. Otherwise

$$\frac{C_{y_1}^2 + \dots + C_{y_v}^2 + v(k-4) + (k-2) + (k-2)(k-3) + C_{k-3}^2}{2k-5+v} \geq k-2,$$

$$y_1^2 + \dots + y_v^2 - 4v + (k-2) + 2(k-2)(k-3) + (k-3)(k-4) \geq 2(k-2)(2k-5)$$

for some natural numbers v, y_1, \dots, y_v such that $y_1 + \dots + y_v = k-2$.

A function $\phi(y_1, \dots, y_{k-2}) = y_1^2 + \dots + y_{k-2}^2 - 4|\{i : y_i \neq 0\}|$ achieves its maximal value on the set $\mathbb{Z}_+^{k-2} \cap \{y_1 + \dots + y_{k-2} = k-2\}$ when $(y_1, \dots, y_{k-2}) = (k-2, 0, \dots, 0)$. Therefore,

$$(k-2)^2 - 4 + (k-2) + 2(k-2)(k-3) + (k-3)(k-4) \geq 2(k-2)(2k-5).$$

A contradiction is obtained. Thus, the vertex \hat{x}_1 is adjacent to $k-1$ or more vertices of the graph $X_j^*(\hat{x}_1)$. If the vertex \hat{x}_1 is not a vertex of this graph, then the density of the graph $X_j^1(\hat{x}_1)|_{V(X_j^*(\hat{x}_1) \cup \{\hat{x}_1\})}$ is greater than the density of the graph $X_j^*(\hat{x}_1)$.

4 Ehrenfeucht game

The main tool in proofs of zero-one laws for the first order properties of the random graphs is a result proved by A. Ehrenfeucht in 1960 (see [16]). In this section we formulate its particular case for graphs. First of all let us define the Ehrenfeucht game on two graphs G, H with i rounds (see [2], [4], [5]–[9], [13], [16]–[12]). Let $V(G) = \{x_1, \dots, x_n\}$, $V(H) = \{y_1, \dots, y_m\}$. At the ν -th step ($1 \leq \nu \leq i$) Spoiler chooses a vertex from any graph. He chooses either a vertex $x_{j_\nu} \in V(G)$ or a vertex $y_{j'_\nu} \in V(H)$. At the same round Duplicator chooses a vertex from the other graph. Let Spoiler choose the vertex $x_{j_\mu} \in V(G)$, $j_\mu = j_\nu$ ($\nu < \mu$), at the μ -th round. Duplicator must choose the vertex $y_{j'_\nu} \in V(H)$. If at this round Spoiler chooses a vertex $x_{j_\mu} \in V(G)$, $j_\mu \notin \{j_1, \dots, j_{\mu-1}\}$, then Duplicator must choose a vertex $y_{j'_\mu} \in V(H)$ such that $j'_\mu \notin \{j'_1, \dots, j'_{\mu-1}\}$. If Duplicator cannot find such a vertex then Spoiler

wins. After the final round vertices $x_{j_1}, \dots, x_{j_i} \in V(G)$, $y_{j'_1}, \dots, y_{j'_i} \in V(H)$ are chosen. Some of these vertices probably coincide. Choose pairwise different vertices: x_{h_1}, \dots, x_{h_l} ; $y_{h'_1}, \dots, y_{h'_l}$, $l \leq i$. Duplicator wins if and only if the corresponding subgraphs are isomorphic:

$$G|_{\{x_{h_1}, \dots, x_{h_l}\}} \cong H|_{\{y_{h'_1}, \dots, y_{h'_l}\}}.$$

Theorem 5 ([16]) *Let G, H be two graphs. Let $i \in \mathbb{N}$ be some natural number. Duplicator has a winning strategy in the game $EHR(G, H, i)$ if and only if for any first-order property L expressed by a formula with quantifier depth at most i either G and H satisfy L or G and H do not satisfy L .*

In the two following sections we state lemmas which we use in the proof of Theorem 1 (see Section 7). We prove lemmas in Section 8.

5 Main lemmas

Let $\mathcal{G} \in \Omega_N$, $\tilde{x} \in \mathcal{V}_N$, $\mathcal{G} \supset Y \supset \mathcal{G}|_{\{\tilde{x}\}}$. We call the pair (Y, \tilde{x}) j -maximal in \mathcal{G} , where $j \in \{1, \dots, \widehat{m}(k)\}$, if $Y = X_j^5(\tilde{x}) = X_{\mathcal{G}}^5(\tilde{x})$.

Let $\mathcal{L}_j^k(N) \subset \Omega_N$ be a set of graphs \mathcal{G} such that there are a vertex \tilde{x} and a graph Y such that the pair (Y, \tilde{x}) is j -maximal in \mathcal{G} . Set

$$\mathcal{L}_{\widehat{m}(k)+1}^k(N) = \Omega_N \setminus (\mathcal{L}_1^k(N) \cup \dots \cup \mathcal{L}_{\widehat{m}(k)}^k(N)),$$

$$\mathcal{A}_{j_1 \dots j_t}^k(N) = \left(\bigcap_{i=1}^t \mathcal{L}_{j_i}^k \right) \cap \left(\Omega_N \setminus \left(\bigcup_{i \in \{1, \dots, \widehat{m}(k)\} \setminus \{j_1, \dots, j_t\}} \mathcal{L}_i^k(N) \right) \right)$$

for any different $j_1, \dots, j_t \in \{1, \dots, \widehat{m}(k) + 1\}$. The following lemma provides pairs of graphs (\tilde{G}, \tilde{H}) such that Duplicator has a winning strategy in the game $EHR(\tilde{G}, \tilde{H}, k)$.

Lemma 1 *For any subset $\{j_1, \dots, j_t\} \subset \{1, \dots, \widehat{m}(k) + 1\}$ Duplicator has a winning strategy in the game $EHR(\tilde{G}, \tilde{H}, k)$ for almost all pairs of graphs (\tilde{G}, \tilde{H}) from $\mathcal{A}_{j_1 \dots j_t}^k(N) \times \mathcal{A}_{j_1 \dots j_t}^k(M)$.*

In the following lemma an asymptotic behavior of probabilities of $\mathcal{A}_{j_1 \dots j_t}^k(N)$ is described.

Lemma 2 *For any different $j_1, \dots, j_t \in \{1, \dots, \widehat{m}(k) + 1\}$ there exist constants $0 \leq \xi_{j_1 \dots j_t} \leq 1$ such that*

$$\lim_{N \rightarrow \infty} P_{N,p}(\mathcal{A}_{j_1 \dots j_t}^k(N)) = \xi_{j_1 \dots j_t}. \quad (1)$$

6 Auxiliary lemmas

In this section we give two statements which we use in the proofs of Lemma 1 and Lemma 2.

Lemma 3 *Let $j_1, \dots, j_l \in \{1, \dots, \widehat{m}(k)\}$ be some numbers (some of them may be equal). Let for any $i_1 \in \{1, \dots, l\}$ there exist $i_2 \in \{1, \dots, l\} \setminus \{i_1\}$ such that the graphs $X_{j_{i_1}}^{**}(\tilde{x}^{i_1}), X_{j_{i_2}}^{**}(\tilde{x}^{i_2})$ have a common vertex. Then $\rho(X_{j_1}^{**}(\tilde{x}^1) \cup \dots \cup X_{j_l}^{**}(\tilde{x}^l)) = k - 2$ if and only if the sets $V(X_{j_i}^{**}(\tilde{x}^i))$ coincide for all $i \in \{1, \dots, l\}$. If not all of the sets coincide then $\rho(X_{j_1}^{**}(\tilde{x}^1) \cup \dots \cup X_{j_l}^{**}(\tilde{x}^l)) > k - 2$.*

For an arbitrary graph G we set $f(G) = v(G) - \alpha \cdot e(G)$.

Lemma 4 *Let G be a strictly balanced graph, $\rho(G) < k - 2$. Let \mathcal{R} be the number of all (K, T) -maximal copies of the graph G in $G(N, p)$ for all α -neutral pairs (K, T) such that $v(T) \leq k^3$, $v(K, T) \leq k^3$. Then \mathcal{R} converges in probability to infinity. The fraction $\frac{\mathcal{R}}{\mathbb{E}_{N,p}\mathcal{R}}$ converges in probability to 1 and $\mathbb{E}_{N,p}\mathcal{R} = \Theta(N^{f(G)})$.*

We prove Lemma 3 and Lemma 4 in Section 8.

7 Proof of Theorem 1

It follows from Lemma 1 that there exists a set $\tilde{\Omega}_N \subset \Omega_N$, $\mathbb{P}_{N,p}(\tilde{\Omega}_N) \rightarrow 1$, $N \rightarrow \infty$, and a partition of this set $\Omega_N^1, \dots, \Omega_N^{s(k)}$, $\bigsqcup_{i=1}^{s(k)} \Omega_N^i = \tilde{\Omega}_N$, such that for any $i \in \{1, \dots, s(k)\}$, $N, M \in \mathbb{N}$ and any pair of graphs $\mathcal{G} \in \tilde{\Omega}_N^i$, $\mathcal{H} \in \tilde{\Omega}_M^i$ Duplicator has a winning strategy in the game $EHR(\mathcal{G}, \mathcal{H}, k)$. Any set Ω_N^i is an intersection of sets $\mathcal{A}_{j_1 \dots j_t}^k(N)$ for some $j_1, \dots, j_t \in \{1, \dots, \widehat{m}(k) + 1\}$ with $\tilde{\Omega}_N \subset \Omega_N$. Let L be a first order property expressed by a formula with a quantifier depth at most k . By Theorem 5 for each $i \in \{1, \dots, s(k)\}$ its truth is the same for all graphs from $\bigcup_{N \in \mathbb{N}} \Omega_N^i$. Lemma 2 provides a convergence of a probability of Ω_N^i for any $i \in \{1, \dots, s(k)\}$. The subset $\mathcal{A}_N(L) \subset \tilde{\Omega}_N$ consisting of all graphs satisfying the property L is the union of Ω_N^i , $i \in I$, for some $I \subseteq \{1, \dots, s(k)\}$. Therefore, $\mathbb{P}_{N,p}(\mathcal{A}_N(L))$ converges too. Theorem is proved.

8 Proofs of lemmas

We do not give a proof of Lemma 4 in the paper because it is a simple version of the proof of Theorem 4 that was proved in [15]. The proof of Lemma 2 is based

on Lemma 3. The proof of Lemma 1 is based on Lemma 4. Therefore, we prove Lemma 3 first and prove Lemma 2 and Lemma 1 after that.

8.1 Proof of Lemma 3

Consider some α -neutral chain (G, H) and graphs $K_1, \dots, K_r, T_1, \dots, T_{r-1}$ such that

- $H = K_1 \subset K_2 \subset \dots \subset K_r = G, T_i \subset K_i, i \in \{1, \dots, r-1\}$,
- the pairs $((K_i \setminus K_{i-1}) \cup T_{i-1}, T_{i-1}), i \in \{2, \dots, r\}$, are α -neutral,
- for any $i \in \{2, \dots, r\}$ the vertices of the graph $K_i \setminus K_{i-1}$ and the vertices of the graph $K_{i-1} \setminus T_{i-1}$ are not adjacent.

Suppose that H is a strictly balanced graph with the density $\rho(H) = 1/\alpha$. Let us prove that the graph G is balanced.

Let F be a proper subgraph of G , $F_1 = F \cap H$, $F_i = F \cap (K_i \setminus K_{i-1}), i \in \{2, \dots, r\}$. From the definition of an α -neutral pair it follows that

$$f(F_i \cup T_{i-1}, T_{i-1}) \geq 0, i \in \{2, \dots, r\}.$$

Obviously,

$$e(F_i \cup \dots \cup F_1, F_{i-1} \cup \dots \cup F_1) \leq e(F_i \cup T_{i-1}, T_{i-1}), i \in \{2, \dots, r\},$$

$$v(F_i \cup \dots \cup F_1, F_{i-1} \cup \dots \cup F_1) = v(F_i \cup T_{i-1}, T_{i-1}), i \in \{2, \dots, r\}.$$

Therefore,

$$f(F_i \cup \dots \cup F_1, F_{i-1} \cup \dots \cup F_1) \geq 0, i \in \{2, \dots, r\}.$$

Furthermore,

$$f(F_1) \geq f(H) = 0$$

as H is a strictly balanced graph. The last inequality is strict if and only if $F_1 \neq \emptyset$. Finally, we get

$$\begin{aligned} \rho(F) &= (k-2) - \frac{(k-2)f(F)}{v(F)} = \\ &= (k-2) - \frac{(k-2)(f(F_1) + \sum_{i=2}^r f(F_i \cup \dots \cup F_1, F_{i-1} \cup \dots \cup F_1))}{v(F)} \leq k-2 = \rho(G). \end{aligned}$$

Therefore, the graph G is balanced.

For each $i \in \{1, \dots, l\}$ denote by Y_i the graph $X_{j_i}^{**}(\tilde{x}_i)$. We prove Lemma 3 by induction. Consider the case $l = 2$. Set $Y_{1,2} = Y_1 \cap Y_2$. Consider the following three situations.

- 1) The set $V(X_{j_1}^*(\tilde{x}^1)) \cap V(Y_{1,2})$ is not empty. The graph $X_{j_1}^*(\tilde{x}^1) \cap Y_{1,2}$ is a proper subgraph of the graph $X_{j_1}^*(\tilde{x}^1)$.
- 2) The set $V(X_{j_1}^*(\tilde{x}^1)) \cap V(Y_{1,2})$ is empty.
- 3) The equality $X_{j_1}^*(\tilde{x}^1) \cap Y_{1,2} = X_{j_1}^*(\tilde{x}^1)$ holds.

The graph $X_{j_1}^*(\tilde{x}^1)$ is strictly balanced with the density equal to $k - 2$. The pair $(Y_1, X_{j_1}^*(\tilde{x}^1))$ is an $1/(k - 2)$ -neutral chain. Therefore, the graph Y_1 is balanced. Thus, in the first case

$$f(Y_1, X_{j_1}^*(\tilde{x}^1) \cup Y_{1,2}) \leq 0, \quad f(Y_2) = 0.$$

Furthermore, the graph $X_{j_1}^*(\tilde{x}^1)$ is strictly balanced, $f(X_{j_1}^*(\tilde{x}^1)) = 0$. Therefore,

$$\frac{e(X_{j_1}^*(\tilde{x}^1)) - e(X_{j_1}^*(\tilde{x}^1), Y_{1,2} \cap X_{j_1}^*(\tilde{x}^1))}{v(X_{j_1}^*(\tilde{x}^1)) - v(X_{j_1}^*(\tilde{x}^1), Y_{1,2} \cap X_{j_1}^*(\tilde{x}^1))} = \frac{e(Y_{1,2} \cap X_{j_1}^*(\tilde{x}^1))}{v(Y_{1,2} \cap X_{j_1}^*(\tilde{x}^1))} < k - 2.$$

So,

$$f(X_{j_1}^*(\tilde{x}^1) \cup Y_{1,2}, Y_{1,2}) \leq f(X_{j_1}^*(\tilde{x}^1), Y_{1,2} \cap X_{j_1}^*(\tilde{x}^1)) < 0. \quad (2)$$

Finally, we get

$$e(Y_1 \cup Y_2) \geq e(Y_2) + e(Y_1, X_{j_1}^* \cup Y_{1,2}) + e(X_{j_1}^*(\tilde{x}^1) \cup Y_{1,2}, Y_{1,2}) > (k - 2)v(Y_1 \cup Y_2).$$

The last inequality is strict due to (2). Thus,

$$\rho(Y_1 \cup Y_2) > k - 2.$$

Consider the second case: $(X_{j_1}^*(\tilde{x}^1) \cap Y_{1,2} = \emptyset)$. From the definition of an α -neutral chain it follows that $v(Y_{1,2}) - \alpha \cdot e(Y_{1,2}) > 0$. Therefore,

$$\begin{aligned} e(Y_1 \cup Y_2) &\geq e(Y_2) + e(Y_1, Y_{1,2}) = e(Y_2) + e(Y_1) - e(Y_{1,2}) > \\ &> (k - 2)(v(Y_2) + v(Y_1) - v(Y_{1,2})) = (k - 2)v(Y_1 \cup Y_2). \end{aligned}$$

We get $\rho(Y_1 \cup Y_2) > k - 2$.

Let finally $Y_{1,2} \supseteq X_{j_1}^*(\tilde{x}^1)$. Then $X_{j_1}^*(\tilde{x}^1) = X_{j_2}^*(\tilde{x}^2)$. Actually if $X_{j_1}^*(\tilde{x}^1) \cap X_{j_2}^*(\tilde{x}^2) \notin \{X_{j_1}^*(\tilde{x}^1), X_{j_2}^*(\tilde{x}^2), \emptyset\}$ then the pair $(X_{j_1}^*(\tilde{x}^1) \cup X_{j_2}^*(\tilde{x}^2), X_{j_1}^*(\tilde{x}^1))$ is α -rigid as the graph $X_{j_2}^*(\tilde{x}^2)$ is strictly balanced. This fact is in conflict with the properties $X_{j_1}^*(\tilde{x}^1) \cup X_{j_2}^*(\tilde{x}^2) \subseteq Y_2$ and $\rho(Y_2) = \rho(X_{j_1}^*(\tilde{x}^1))$ as the graph Y_2 is balanced. In the cases $X_{j_1}^*(\tilde{x}^1) \subset X_{j_2}^*(\tilde{x}^2)$, $X_{j_2}^*(\tilde{x}^2) \subset X_{j_1}^*(\tilde{x}^1)$ we also get rigid pairs and obtain contradiction. If $X_{j_1}^*(\tilde{x}^1) \cap X_{j_2}^*(\tilde{x}^2) = \emptyset$ then the graph $Y_2 \setminus X_{j_2}^*(\tilde{x}^2)$ contains the subgraph $X_{j_1}^*(\tilde{x}^1)$ with the density $k - 2$. It is impossible since $(Y_2, X_{j_2}^*(\tilde{x}^2))$ is α -neutral chain.

Consider $l \geq 3$ pairs (Y_i, \tilde{x}^i) . Let $V(Y_1) \cap V(Y_2) \neq \emptyset$, $V(X_{j_1}^*(\tilde{x}^1)) \neq V(X_{j_2}^*(\tilde{x}^2))$,

$$\rho(Y_1 \cup \dots \cup Y_{l-1}) > k - 2.$$

The graph $\bigcup_{i=1}^{l-1} Y_i \cap Y_l$ is a subgraph of the graph Y_l . We have proved that Y_l is a balanced graph. Therefore,

$$\rho\left(\bigcup_{i=1}^{l-1} Y_i \cap Y_l\right) \leq k - 2.$$

Thus,

$$\rho\left(\bigcup_{i=1}^l Y_i\right) = \frac{e\left(\bigcup_{i=1}^l Y_i\right)}{v\left(\bigcup_{i=1}^l Y_i\right)} \geq \frac{e\left(\bigcup_{i=1}^{l-1} Y_i\right) + e(Y_l) - e\left(\bigcup_{i=1}^{l-1} Y_i \cap Y_l\right)}{v\left(\bigcup_{i=1}^{l-1} Y_i\right) + v(Y_l) - v\left(\bigcup_{i=1}^{l-1} Y_i \cap Y_l\right)} > k - 2.$$

So, the density equals $k - 2$ if and only if for any graphs Y_{i_1}, Y_{i_2} with common vertices $X_{j_{i_1}}^*(\tilde{x}^{i_1}) = X_{j_{i_2}}^*(\tilde{x}^{i_2})$. For any $i_1 \in \{1, \dots, l\}$ there exists $i_2 \in \{1, \dots, l\} \setminus \{i_1\}$ such that the graphs Y_{i_1}, Y_{i_2} have a common vertex. Therefore, the density equals $k - 2$ if and only if the sets $V(X_{j_i}^*(\tilde{x}^i))$ coincide for all $i \in \{1, \dots, l\}$. Lemma is proved.

8.2 Proof of Lemma 2

Let us prove the convergence of $P_{N,p}(\mathcal{L}_j^k(N))$ to some number ξ_j for each $j \in \{1, \dots, m(k)\}$ as $N \rightarrow \infty$. The proof is based on three statements. The first one,

Lemma 3, is already proved. The second one is stated and proved in [12]. An analogue of the third statement is proved there too. Let us introduce some notation.

Let $j \in \{1, \dots, m(k)\}$. Let v_j and e_j be the numbers of vertices and edges in the graph $X_j^5(\widehat{x}_1)$ respectively. Let a_j be the number of automorphisms of the graph $X_j^5(\widehat{x}_1)$ with the fixed point \widehat{x}_1 . Consider all ordered collections of v_j vertices of the set \mathcal{V}_N . Let us define a subset M_j of the set of all such collections.

- M_j contains all different unordered collections.
- Let $(\tilde{x}_{i_1}, \dots, \tilde{x}_{i_{v_j}}) \in M_j$. Let \tilde{Y} be a graph on the set of vertices $\{\tilde{x}_{i_1}, \dots, \tilde{x}_{i_{v_j}}\}$. Assume that \tilde{Y} is a strict $(X_j^5(\widehat{x}_1), \widehat{G}|_{\{\widehat{x}_1\}})$ -extension of the graph $\tilde{Y}|_{\{\tilde{x}_{i_1}\}}$. Let a graph obtained by permutation of vertices $\begin{pmatrix} i_2 & \dots & i_{v_j} \\ t_2 & \dots & t_{v_j} \end{pmatrix}$ of the graph \tilde{Y} be a strict $(X_j^5(\widehat{x}_1), \widehat{G}|_{\{\widehat{x}_1\}})$ -extension of the graph $\tilde{Y}|_{\{\tilde{x}_{i_1}\}}$. Then the set M_j does not contain the collection $(\tilde{x}_{i_1}, \tilde{x}_{t_2}, \dots, \tilde{x}_{t_{v_j}})$. Otherwise this collection is in M_j .
- In M_j there are no collections except the described ones.

Set $m_j = |M_j|$. Let us enumerate all the collections from the set M_j by numbers $1, \dots, m_j$. Consider events $\mathcal{B}_1^j, \dots, \mathcal{B}_{m_j}^j$. The event \mathcal{B}_i^j is that a subgraph \tilde{Y}_i on the i -th collection from M_j and its first vertex form a j -maximal pair. Let A_i^j be an indicator of the event \mathcal{B}_i^j . Consider a random variable $A_j = \sum_{i=1}^{m_j} A_i^j$ equal to a number of all j -maximal pairs. We get

$$\begin{aligned} \mathbb{P}_{N,p}(A_j = 0) &= 1 - \sum_{i=1}^{m_j} \mathbb{P}_{N,p}(\mathcal{B}_i^j) + \sum_{i_1, i_2=1}^{m_j} \mathbb{P}_{N,p}(\mathcal{B}_{i_1}^j \cap \mathcal{B}_{i_2}^j) + \dots + \\ &+ (-1)^n \sum_{i_1, i_2, \dots, i_n=1}^{m_j} \mathbb{P}_{N,p}(\mathcal{B}_{i_1}^j \cap \mathcal{B}_{i_2}^j \cap \dots \cap \mathcal{B}_{i_n}^j) + \dots \end{aligned} \quad (3)$$

The summation is over all different collections with pairwise different numbers. Let us prove that there exists a number ξ_j such that

$$\lim_{N \rightarrow \infty} \mathbb{P}_{N,p}(\mathcal{L}_j^k(N)) = \lim_{N \rightarrow \infty} (1 - \mathbb{P}_{N,p}(A_j = 0)) = \xi_j.$$

Let $\phi_1^j(N)$ be the probability that the pair $(\tilde{Y}, \tilde{x}_{i_1})$, $\tilde{Y} = \mathcal{G}|_{\{\tilde{x}_{i_1}, \dots, \tilde{x}_{i_{v_j}}\}}$, $\mathcal{G} \in \Omega_N$, is j -maximal under the condition that the graph \tilde{Y} is a strict $(X_j^5(\hat{x}_1), \hat{G}|_{\{\hat{x}_1\}})$ -extension of the graph $\tilde{Y}|_{\{\tilde{x}_1\}}$. Then

$$\sum_{i=1}^{m_j} \mathbb{P}_{N,p}(\mathcal{B}_i^j) = \mathbb{E}_{N,p}(A_j) = NC_{N-1}^{v_j-1} \frac{(v_j-1)!}{a_j} \phi_1^j(N) p^{e_j} \sim \frac{\phi_1^j(N)}{a_j},$$

where $\mathbb{E}_{N,p}(X_j) = \mathbb{E}(X_j(G(N, p)))$. Set

$$a_n^j(N) = \sum_{i_1, i_2, \dots, i_n=1}^{m_j} \mathbb{P}_{N,p}(\mathcal{B}_{i_1}^j \cap \mathcal{B}_{i_2}^j \cap \dots \cap \mathcal{B}_{i_n}^j).$$

We use the notation $i_1 \sim i_2$ in the following case: the numbers i_1, i_2 are from $\{1, \dots, m_j\}$, $i_1 \neq i_2$, and the collections from M_j numerated by i_1, i_2 have common vertices. Denote the sum with intersecting collections of vertices by $r_j(n, N)$. In other words

$$a_n^j(N) - \sum_{i_1, i_2, \dots, i_n: \forall t_1 \neq t_2 \in \{1, \dots, n\} \ i_{t_1} \neq i_{t_2}, i_{t_1} \sim i_{t_2}} \mathbb{P}_{N,p}(\mathcal{B}_{i_1}^j \cap \mathcal{B}_{i_2}^j \cap \dots \cap \mathcal{B}_{i_n}^j) = r_j(n, N).$$

Let $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ be pairwise disjoint collections from \mathcal{V}_N with cardinality v_j . Let \tilde{x}_t^i be a vertex numerated by t in the i -th collection, $i \in \{1, \dots, n\}$, $t \in \{1, \dots, v_j\}$. Let $\mathcal{Y}_n^j(N)$ be a set of all graphs \mathcal{G} from Ω_N such that for any $i \in \{1, \dots, n\}$ the pair $(\mathcal{G}|_{\mathbf{Y}_i}, \tilde{x}_1^i)$ is j -maximal in \mathcal{G} . Denote by $\mathcal{X}_n^j(N)$ a set of all graphs \mathcal{G} in Ω_N such that the subgraphs $\mathcal{G}|_{\mathbf{Y}_1}, \dots, \mathcal{G}|_{\mathbf{Y}_n}$ are strict $(X_j^5(\hat{x}_1), \hat{G}|_{\{\hat{x}_1\}})$ -extensions of graphs $\mathcal{G}|_{\{\tilde{x}_1^1\}}, \dots, \mathcal{G}|_{\{\tilde{x}_1^n\}}$ respectively. Let

$$\phi_n^j(N) = \mathbb{P}_{N,p}(\mathcal{Y}_n^j(N) | \mathcal{X}_n^j(N)).$$

Obviously the probability $\phi_n^j(N)$ does not depend on a choice of sets $\mathbf{Y}_1, \dots, \mathbf{Y}_n$. We get

$$a_n^j(N) = \sum_{i_1, i_2, \dots, i_n=1}^{m_j} \mathbb{P}_{N,p}(\mathcal{B}_{i_1}^j \cap \mathcal{B}_{i_2}^j \cap \dots \cap \mathcal{B}_{i_n}^j) \sim \frac{\phi_n^j(N)}{n!} \left(\frac{1}{a_j}\right)^n + r_j(n, N).$$

Let us formulate a statement from [12] (see Statement 3).

Statement 1 *Let $\{a_n(N)\}_{n \in \mathbb{N}}$ be a set of functions such that there exists a sequence of numbers $\{b_n\}_{n \in \mathbb{N}}$ obeying the following law: $\forall n \in \mathbb{N} \ a_n(N) \rightarrow b_n, N \rightarrow \infty$. Let $\sum_{n=1}^{\infty} b_n = b$. If for any $N \in \mathbb{N}$ the series $\sum_{n=1}^{\infty} a_n(N)$ converges and for every $s \in \mathbb{N}$, $N \in \mathbb{N}$*

$$\sum_{n=1}^{2s-1} a_n(N) \leq \sum_{n=1}^{\infty} a_n(N) \leq \sum_{n=1}^{2s} a_n(N),$$

then $\sum_{n=1}^{\infty} a_n(N) \rightarrow b, N \rightarrow \infty$.

For any $n \in \mathbb{N}$, $N \in \mathbb{N}$ the inequality $a_n^j(N) \geq a_{n+1}^j(N)$ holds. Therefore by Statement 1 the convergence

$$P_{N,p}(\mathcal{L}_j^k(N)) \rightarrow \xi_j$$

follows from the following fact. For each $n \in \mathbb{N}$

$$\lim_{N \rightarrow \infty} (a_n^j(N) - r_j(n, N)) = b_j(n), \quad (4)$$

$$\lim_{N \rightarrow \infty} r_j(n, N) = r_j(n), \quad (5)$$

$$\sum_{n=1}^{\infty} (-1)^n (b_j(n) + r_j(n)) < \infty. \quad (6)$$

The equality (4) follows from a statement similar to Statement 2 from [12]. We do not give here a proof of the statement because the proofs of these two statements are the same.

Statement 2 *There exists $0 < \zeta_j < 1$ such that*

$$\phi_1^j(N) \sim \zeta_j, \quad \phi_n^j(N) \sim \zeta_j^n.$$

All that remains is to show that equalities (5) and

$$\lim_{n \rightarrow \infty} r_j(n) = 0$$

hold.

Let x be a vertex. Let us define sets $\mathcal{Q}_1^n(x), \mathcal{Q}_2^n(x)$ in the following way:

$$\begin{aligned} (Q, x) \in \mathcal{Q}_1^n &\Leftrightarrow ((v_j < v(Q) < nv_j) \wedge (\rho(Q) = k - 2) \wedge (\exists \hat{Y}_1, \dots, \hat{Y}_n (\forall i \in \{1, \dots, n\} \\ ((\hat{Y}_i, x) \cong (\hat{Y}_j(\hat{x}), \hat{x})) \wedge (\forall i_1, i_2 \in \{1, \dots, n\} (\hat{Y}_{i_1} \cap \hat{Y}_{i_2} \supset X_j^*(x))) \wedge (Q = \hat{Y}_1 \cup \dots \cup \hat{Y}_n))), \\ (Q, x) \in \mathcal{Q}_2^n &\Leftrightarrow ((v_j < v(Q) < nv_j) \wedge (\rho(Q) > k - 2) \wedge (\exists \hat{Y}_1, \dots, \hat{Y}_n \\ \forall i_1 \in \{1, \dots, n\} ((\hat{Y}_{i_1} \cong \hat{Y}_j(\hat{x})) \wedge (\exists i_2 \in \{1, \dots, n\} (\hat{Y}_{i_1} \cap \hat{Y}_{i_2} \neq \emptyset))) \wedge (Q = \hat{Y}_1 \cup \dots \cup \hat{Y}_n))). \end{aligned}$$

Let $(Q_1, x) \in \mathcal{Q}_{i_1}^{r_1}, \dots, (Q_t, x) \in \mathcal{Q}_{i_t}^{r_t}, i_1, \dots, i_t \in \{1, 2\}, r_1, \dots, r_t \in \mathbb{N}, r_1 + \dots + r_t = n$. Let $\hat{Y}_1^l \cup \dots \cup \hat{Y}_{r_l}^l$ be a decomposition of $Q_l, l \in \{1, \dots, t\}$, into graphs isomorphic to $X_j^5(\hat{x})$. Let us introduce different collections of vertices from \mathcal{V}_N for graphs $Q_l, l \in \{1, \dots, t\}$, in the same way as M_j was introduced. The first vertex in a collection is fixed if and only if $(Q_l, x) \in \mathcal{Q}_1^{r_l}$. For every $l \in \{1, \dots, t\}$ define an event $\mathcal{B}_i(Q_l)$. Its definition depends on whether $(Q_l, x) \in \mathcal{Q}_1^{r_l}$ or $(Q_l, x) \in \mathcal{Q}_2^{r_l}$. If $(Q_l, x) \in \mathcal{Q}_2^{r_l}$ then the event $\mathcal{B}_i(Q_l)$ is that the subgraph induced on the i -th collection is isomorphic to Q_l . If $(Q_l, x) \in \mathcal{Q}_1^{r_l}$ then the event $\mathcal{B}_i(Q_l)$ is that the subgraph induced on the i -th collection is a strict $(Q_l, \{x\})$ -extension of the first vertex of the collection and forms with it a j -maximal pair. By Lemma 3 for any $Q_i \in \mathcal{Q}_{l_i}^{r_i}, i \in \{1, \dots, t\}$, there exist numbers $q(Q_1, \dots, Q_t) > 0$ such that

$$\begin{aligned} \sum_{r_1 + \dots + r_t = n} \sum_{i=1}^t \sum_{(Q_i, x) \in \mathcal{Q}_1^{r_i}} q(Q_1, \dots, Q_t) \sum_{i_1, \dots, i_t} \mathbb{P}_{N,p}(\mathcal{B}_{i_1}(Q_1) \cap \dots \cap \mathcal{B}_{i_t}(Q_t)) &\leq r_j(n, N) \leq \\ &\leq \sum_{r_1 + \dots + r_t = n} \sum_{i=1}^t \sum_{l_i=1}^2 \sum_{(Q_i, x) \in \mathcal{Q}_{l_i}^{r_i}} q(Q_1, \dots, Q_t) \sum_{i_1, \dots, i_t} \mathbb{P}_{N,p}(\mathcal{B}_{i_1}(Q_1) \cap \dots \cap \mathcal{B}_{i_t}(Q_t)). \quad (7) \end{aligned}$$

Summations in 7 are over i_1, \dots, i_t corresponding to pairwise disjoint collections.

Let $r_1, \dots, r_t \in \mathbb{N}, r_1 + \dots + r_t = n$. Consider a vector $(l_1, \dots, l_t) \in \{1, 2\}^t$ such that at least one of the numbers l_1, \dots, l_t equals 2. Let $(Q_i, x) \in \mathcal{Q}_{l_i}^{r_i}$. Consider graphs $\hat{Q}_1, \dots, \hat{Q}_t$ with the following properties.

- Any two graphs among $\hat{Q}_1, \dots, \hat{Q}_t$ do not have a common vertex.
- There exist vertices x_1, \dots, x_t such that $(\hat{Q}_i, x_i) \cong (Q_i, x)$ for any $i \in \{1, \dots, t\}$.

Set $V(G) = V(\widehat{Q}_1) \cup \dots \cup V(\widehat{Q}_t)$, $E(G) = E(\widehat{Q}_1) \cup \dots \cup E(\widehat{Q}_t)$. Let N_G be the number of copies of G in $G(N, p)$. Obviously there exist $C(G), \mu(G) > 0$ such that

$$E_{N,p} N_G < C(G) N^{v(G) - \alpha \cdot e(G)} < C(G) N^{-\mu(G)}.$$

Therefore the difference between the upper and the lower bound in (7) equals $o(1)$. The existence of $\lim_{N \rightarrow \infty} r_j(n, N)$ follows from the convergence of the lower bound in (7) to a number $r_j(n)$ as $N \rightarrow \infty$. Let us prove this convergence. Let R_1, \dots, R_t be pairwise disjoint subsets of V_N , $|R_i| = |V(Q_i)|$. Let $\varphi(Q_1, \dots, Q_t)$ be the probability that the graphs induced on R_1, \dots, R_t form with the first vertices of R_1, \dots, R_t j -maximal pairs under the following condition. For each $l \in \{1, \dots, t\}$ the graph induced on the l -th collection is a strict (Q_l, x) -extension of the first vertex of this collection. The proof of the convergence of $\varphi(Q_1, \dots, Q_t)$, $N \rightarrow \infty$, is identical to the proof of Statement 2. Thus we do not give this proof. The existence of $r_j(n)$ follows from the convergence of $\varphi(Q_1, \dots, Q_t)$.

Finally let us prove that $r_j(n) \rightarrow 0$ when $n \rightarrow \infty$. It is easy to see that

$$\begin{aligned} & \sum_{r_1 + \dots + r_t = n} \sum_{i=1}^t \sum_{(Q_i, x) \in \mathcal{Q}_1^{r_i}} q(Q_1, \dots, Q_t) \sum_{i_1, \dots, i_t} P_{N,p}(\mathcal{B}_{i_1}(Q_1) \cap \dots \cap \mathcal{B}_{i_t}(Q_t)) \\ & \leq (C_N^{v_j})^n \left(\frac{v_j!}{a_j} \right)^n \frac{1}{n!} p^{e(Q_1) + \dots + e(Q_t)} \varphi(Q_1, \dots, Q_t) \leq \frac{1}{a_j^n n!} = o(1). \end{aligned}$$

Therefore the convergence of $P_{N,p}(\mathcal{L}_j^k(N))$ is proved.

Let us consider intersections of the properties $\mathcal{L}_j^k(N)$.

The convergence (1) follows from the existence of a limit of the sequence $\{P_{N,p}(\mathcal{L}_{j_1}^k(N) \cap \dots \cap \mathcal{L}_{j_t}^k(N))\}_{N \in \mathbb{N}}$ for any $j_1, \dots, j_t \in \{1, \dots, m(k)\}$. Indeed, for any properties A, C the equality $P(\overline{A} \cap C) = P(C) - P(A \cap C)$ holds. If $P(\overline{A_1} \cap \dots \cap \overline{A_{k-1}} \cap C)$ equals

$$\sum_s \sum_{i_1, \dots, i_s} (-1)^{\sigma_s(i_1, \dots, i_s)} P(A_{i_1} \cap \dots \cap A_{i_s} \cap C)$$

for some $\sigma_s : \mathbb{N}^s \rightarrow \{0, 1\}$, then

$$\begin{aligned} P(\overline{A_1} \cap \dots \cap \overline{A_k} \cap C) &= P(\overline{A_1} \cap \dots \cap \overline{A_{k-1}} \cap C) - P(\overline{A_1} \cap \dots \cap (A_k \cap C)) = \\ &= \sum_s \sum_{i_1, \dots, i_s} (-1)^{\sigma(i_1, \dots, i_s)} (P(A_{i_1} \cap \dots \cap A_{i_s} \cap C) - P(A_{i_1} \cap \dots \cap A_{i_s} \cap A_k \cap C)). \end{aligned}$$

In other words, the probability $P(\overline{A_1} \cap \dots \cap \overline{A_k} \cap C)$ can be written as the finite sum of the probabilities of some intersections of properties without any negations. Therefore the existence of a limit of any such intersection implies the existence of a limit of $P(\overline{A_1} \cap \dots \cap \overline{A_k} \cap C)$.

Thus we have $j_1, \dots, j_t \in \{1, \dots, m(k)\}$. The proof of the existence of $\lim_{n \rightarrow \infty} P_{N,p}(\mathcal{L}_{j_1}^k(N) \cap \dots \cap \mathcal{L}_{j_t}^k(N))$ and the proof of the convergence of the probability of one property are the same. Note that if an intersection of $\{j_1, \dots, j_t\}$ and $\{m(k) + 1, \dots, \widehat{m}(k)\}$ is not empty, then the probability of the existence of $X_{j_i}^{**}(\tilde{x}_i)$ converges due to arguments which are the same as in the case $j_1, \dots, j_t \in \{1, \dots, m(k)\}$. Therefore it remains to apply Theorem 4. Finally the convergence $P_{N,p}(\mathcal{L}_{\widehat{m}(k)+1}^k(N))$ follows from the equality

$$\mathcal{L}_{\widehat{m}(k)+1}^k(N) = \Omega_N \setminus \mathcal{L}_1^k(N) \cup \dots \cup \mathcal{L}_{\widehat{m}(k)}^k(N).$$

Lemma is proved .

8.3 Proof of Lemma 1

Let \mathcal{S} be the set of all α -rigid and α -neutral pairs (K, T) such that $v(T) \leq k^3, v(K, T) \leq k^3$. Theorem 3, Theorem 4, Lemma 4 imply the existence of a set $\tilde{\Omega}_N \subset \Omega_N$ such that

$$\lim_{N \rightarrow \infty} P_{N,p}(\Omega_N \setminus \tilde{\Omega}_N) = 0$$

and the following property holds. For any $\mathcal{G} \in \tilde{\Omega}_N$, $r \leq k$, $\tilde{x}_1, \dots, \tilde{x}_r$, $(K, T) \in \mathcal{S}$ there exist all possible non-isomorphic (K, T) -maximal α -safe pairs $(W, \mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_r\}})$, $v(W) \leq k^3$, in \mathcal{G} and all possible non-isomorphic (K, T) -maximal in \mathcal{G} strictly balanced graphs W with $\rho(W) < \alpha$, $v(W) \leq k^3$, and there is no copy of a graph with $r \leq k^3$ vertices and density greater than α .

If in some rounds a strategy of Spoiler doesn't depend on the choice between the graphs \mathcal{G} and \mathcal{H} then we assume that he chooses the graph \mathcal{G} .

Let us prove that for any $N, M \in \mathbb{N}$ and any pair

$$(\mathcal{G}, \mathcal{H}) \in (\mathcal{A}_{j_1 \dots j_t}(N) \cap \tilde{\Omega}_N) \times (\mathcal{A}_{j_1 \dots j_t}(M) \cap \tilde{\Omega}_M)$$

Duplicator has a winning strategy in the game $EHR(\mathcal{G}, \mathcal{H}, k)$.

Let Spoiler choose a vertex \tilde{x}_1 in \mathcal{G} at the first round. Consider the graph $X_{\mathcal{G}}^5(\tilde{x}_1)$. If $\rho^{\max}(X_{\mathcal{G}}^5(\tilde{x}_1)) = k - 2$ and $X_{\mathcal{G}}^5(\tilde{x}_1)$ is (\tilde{x}_1) -net in \mathcal{G} , then as $\mathcal{H} \in \mathcal{A}_{j_1 \dots j_t}(M)$ there is a vertex \tilde{y}_1 in \mathcal{H} such that $X_{\mathcal{H}}^5(\tilde{y}_1)$ is a net of the graph $X_{\mathcal{G}}^5(\tilde{x}_1)$ (in the considered case graphs $X_{\mathcal{G}}^5(\tilde{x}_1)$ and $X_{\mathcal{H}}^5(\tilde{y}_1)$ are isomorphic). Let either $\rho^{\max}(X_{\mathcal{G}}^5(\tilde{x}_1)) = k - 2$ and $X_{\mathcal{G}}^5(\tilde{x}_1)$ be not a (\tilde{x}_1) -net in \mathcal{G} or $\rho^{\max}(X_{\mathcal{G}}^5(\tilde{x}_1)) < k - 2$. As $\mathcal{H} \in \tilde{\Omega}_M$ in the graph \mathcal{H} there is a vertex \tilde{y}_1 such that the graph $X_{\mathcal{H}}^5(\tilde{y}_1)$ is a net of the graph $X_{\mathcal{G}}^5(\tilde{x}_1)$. Duplicator chooses the vertex \tilde{y}_1 at the first round.

Let at the ξ -th round, $\xi \in \{2, \dots, k - 3\}$, Spoiler choose a vertex $\tilde{x}_{\xi} \in \mathcal{G}$. If for some $i \in \{1, t(\tilde{x}_1, \dots, \tilde{x}_{\xi-1}, \mathcal{G})\}$, $\mu \in \{\xi, \dots, k - 3\}$ vertices \tilde{x}_{ξ} and $\tilde{x}_{\mu}^i(\tilde{x}_1, \dots, \tilde{x}_{\xi-1})$ coincide then Duplicator chooses the vertex

$$\tilde{y}_{\xi} = \text{NET}_{\mathcal{G}, \mathcal{H}, \tilde{x}_1, \dots, \tilde{x}_{\xi-1}, \tilde{y}_1, \dots, \tilde{y}_{\xi-1}}(\tilde{x}_{\mu}^i(\tilde{x}_1, \dots, \tilde{x}_{\xi-1})).$$

Suppose there are no appropriate $i \in \{1, t(\tilde{x}_1, \dots, \tilde{x}_{\xi-1}, \mathcal{G})\}$ and $\mu \in \{\xi, \dots, k - 3\}$. As $\mathcal{H} \in \Omega_M$ from the definitions of $X_{\mathcal{G}}^5(\tilde{x}_1, \dots, \tilde{x}_{\xi-1})$, $X_{\mathcal{G}}^5(\tilde{x}_1, \dots, \tilde{x}_{\xi})$ it follows that in the graph \mathcal{H} there is a vertex \tilde{y}_{ξ} such that the graph $X_{\mathcal{H}}^5(\tilde{y}_1, \dots, \tilde{y}_{\xi})$ is a net of the graph $X_{\mathcal{G}}^5(\tilde{x}_1, \dots, \tilde{x}_{\xi})$. Indeed, we want to construct a graph $X_{\mathcal{H}}^5(\tilde{y}_1, \dots, \tilde{y}_{\xi})$ such that the pair $(X_{\mathcal{H}}^5(\tilde{y}_1, \dots, \tilde{y}_{\xi}), \mathcal{H}|_{\{\tilde{y}_1, \dots, \tilde{y}_{\xi-1}\}})$ is α -safe. Duplicator chooses the vertex \tilde{y}_{ξ} .

Let at the $k - 3$ -th round vertices $\tilde{x}_1, \dots, \tilde{x}_{k-3} \in \mathcal{G}$, $\tilde{y}_1, \dots, \tilde{y}_{k-3} \in \mathcal{H}$ be chosen. The graphs $X_{\mathcal{G}}^5(\tilde{x}_1, \dots, \tilde{x}_{k-3})$, $X_{\mathcal{H}}^5(\tilde{y}_1, \dots, \tilde{y}_{k-3})$ are isomorphic (it follows from the ways of their constructions). Let $\varphi : X_{\mathcal{G}}^5(\tilde{x}_1, \dots, \tilde{x}_{k-3}) \rightarrow X_{\mathcal{H}}^5(\tilde{y}_1, \dots, \tilde{y}_{k-3})$ be an isomorphism.

The remaining part of the proof is divided into cases. There are some basic cases such that other cases are similar to them. Thus we give proofs for the basic cases only. However we give brief proofs for all the cases.

1. At the $k - 2$ -th round Spoiler chooses a vertex \tilde{x}_{k-2} adjacent to $\tilde{x}_1, \dots, \tilde{x}_{k-3}$.

If in \mathcal{G} there are vertices \tilde{x}^1, \tilde{x}^2 adjacent to each of the vertices $\tilde{x}_1, \dots, \tilde{x}_{k-2}$, then the vertex \tilde{x}_{k-2} is in $V(\hat{X}_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$. Duplicator chooses $\tilde{y}_{k-2} = \varphi(\tilde{x}_{k-2})$. If at the $k - 1$ -th round Spoiler chooses a vertex from $V(\hat{X}_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$, then Duplicator chooses the vertex $\tilde{y}_{k-1} = \varphi(\tilde{x}_{k-1})$ again and obviously wins. If Spoiler chooses a vertex \tilde{x}_{k-1} adjacent to each of the vertices $\tilde{x}_1, \dots, \tilde{x}_{k-2}$ and there is no vertex \tilde{x} adjacent to $\tilde{x}_1, \dots, \tilde{x}_{k-1}$, then without loss of generality one can consider \tilde{x}_{k-1} to be an element of the set $X_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3})$. Spoiler chooses the vertex $\tilde{y}_{k-1} = \varphi(\tilde{x}_{k-1})$. If at the k -th round Spoiler chooses a vertex \tilde{x}_k adjacent to $k - 2$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-1}$ (say, to vertices $\tilde{x}_1, \dots, \tilde{x}_{k-2}$), then in $X_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3})$ there is a vertex \tilde{x} , adjacent to $\tilde{x}_1, \dots, \tilde{x}_{k-2}$. Duplicator chooses $\tilde{y}_k = \varphi(\tilde{x}_k)$ and wins. Finally, if the vertex \tilde{x}_k is adjacent to at most

$k - 3$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-1}$, then the pair $(\mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_k\}}, \mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_{k-1}\}})$ is α -safe. As $\mathcal{H} \in \tilde{\Omega}_M$, in \mathcal{H} there is a vertex \tilde{y}_k such that the graph $\mathcal{H}|_{\{\tilde{y}_1, \dots, \tilde{y}_k\}}$ is a strict $(\mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_k\}}, \mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_{k-1}\}})$ -extension of the graph $\mathcal{H}|_{\{\tilde{y}_1, \dots, \tilde{y}_{k-1}\}}$. Duplicator chooses \tilde{y}_k and wins. Let the vertex \tilde{x}_{k-1} be adjacent to at most $k - 4$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-2}$. Let $\tilde{x}^1, \dots, \tilde{x}^s$ be all vertices from \mathcal{G} adjacent to at most $k - 2$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-1}$ such that pairs $(\mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_{k-1}, \tilde{x}^i\}}, \mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_{k-1}\}})$, $i \in \{1, \dots, s\}$, are non-isomorphic. Consider a subgraph A of \mathcal{G} containing the vertices $\tilde{x}_1, \dots, \tilde{x}_{k-1}, \tilde{x}^1, \dots, \tilde{x}^s$ only. The pair $(A, \mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_{k-2}\}})$ is α -safe. As $\mathcal{H} \in \tilde{\Omega}_M$, in \mathcal{H} there is a strict $(A, \mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_{k-2}\}})$ -extension B of the graph $\mathcal{H}|_{\{\tilde{y}_1, \dots, \tilde{y}_{k-3}\}}$. Let $\xi : A \rightarrow B$ be an isomorphism corresponding to this extension. Duplicator chooses $\xi(\tilde{x}_{k-1})$ and wins. Lastly, let the vertex \tilde{x}_{k-1} be adjacent to $k - 3$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-2}$. If there is a vertex \tilde{x} adjacent to each of $\tilde{x}_1, \dots, \tilde{x}_{k-1}$, then without loss of generality one can consider vertices $\tilde{x}_{k-1}, \tilde{x}$ to be in $V(X_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$. Duplicator chooses a vertex $\tilde{y}_{k-1} = \varphi(\tilde{x}_{k-1})$ and wins. If Spoiler chooses a vertex adjacent to each of $\tilde{x}_1, \dots, \tilde{x}_{k-1}$, then Duplicator chooses $\varphi(\tilde{x})$. Let Spoiler choose a vertex adjacent to $k - 2$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-1}$. We can assume that this vertex is from $V(X_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$. Therefore, Duplicator wins again.

Let \tilde{x}_{k-2} be not from $V(\hat{X}_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$. Let $\tilde{x}^1, \dots, \tilde{x}^s$ be all vertices of the graph \mathcal{G} such that the following properties hold. Each of the vertices $\tilde{x}_1, \dots, \tilde{x}_{k-2}$ is adjacent to each of the vertices $\tilde{x}^1, \dots, \tilde{x}^s$. Sets of collections containing $k - 3$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-2}$ such that in \mathcal{G} there is a vertex adjacent to them and to \tilde{x}^i are different for all $i \in \{1, \dots, s\}$. Consider a subgraph A of \mathcal{G} containing the vertices $\tilde{x}_1, \dots, \tilde{x}_{k-2}, \tilde{x}^1, \dots, \tilde{x}^s$ and one (G_2, H_2) -extension for each subgraph with $k - 3$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-2}$ and one from $\tilde{x}^1, \dots, \tilde{x}^s$. Then the pair $(A, \mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_{k-3}\}})$ is α -safe. As $\mathcal{H} \in \tilde{\Omega}_M$ in \mathcal{H} there is a strict $(A, \mathcal{G}|_{\{\tilde{x}_1, \dots, \tilde{x}_{k-3}\}})$ -extension B of the graph $\mathcal{H}|_{\{\tilde{y}_1, \dots, \tilde{y}_{k-3}\}}$. Let $\xi : A \rightarrow B$ be an isomorphism corresponding to this extension. Duplicator chooses $\tilde{y}_{k-2} = \xi(\tilde{x}_{k-2})$. Let at the $k - 1$ -th round Spoiler choose a vertex \tilde{x}_{k-1} adjacent to each of $\tilde{x}_1, \dots, \tilde{x}_{k-2}$. There is such a vertex in B that Duplicator can win by choosing this vertex. If Spoiler chooses a vertex \tilde{x}_{k-1} adjacent to at most $k - 4$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-2}$, then obviously Duplicator has a winning strategy. Finally, let \tilde{x}_{k-1} be adjacent to $k - 3$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-2}$. If there is a vertex \tilde{x} adjacent to each of $\tilde{x}_1, \dots, \tilde{x}_{k-1}$, then without loss of generality we can consider vertices $\tilde{x}_{k-1}, \tilde{x}$ to be in $V(A)$. Then there is a vertex \tilde{y}_{k-1} in B , corresponding to the vertex \tilde{x}_{k-1} . Duplicator chooses this vertex and wins.

2. At the $k - 2$ -th round Spoiler chooses a vertex \tilde{x}_{k-2} adjacent to $k - 4$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-3}$.

If in \mathcal{G} there are vertices \tilde{x}^1, \tilde{x}^2 , adjacent to each of $\tilde{x}_1, \dots, \tilde{x}_{k-2}$, then either the vertices \tilde{x}^1, \tilde{x}^2 are in $V(\hat{X}_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$, or the vertices \tilde{x}^1, \tilde{x}^2 are in $V(\hat{X}_{\mathcal{G}}^4(\tilde{x}_1, \dots, \tilde{x}_{k-3})) \setminus V(\hat{X}_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$. Anyway the vertex \tilde{x}_{k-2} is in $V(\hat{X}_{\mathcal{G}}^4(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$. Spoiler chooses $\tilde{y}_{k-2} = \varphi(\tilde{x}_{k-2})$. Further choices of Duplicator are described in the same manner as for the case 1.

If there are no two vertices adjacent to each other and to each of $\tilde{x}_1, \dots, \tilde{x}_{k-2}$, then further reasonings are identical to reasonings from subcases of the case 1., in which we use safe pairs.

3. At the $k - 2$ -th round Spoiler chooses a vertex \tilde{x}_{k-2} , adjacent to at most $k - 5$ vertices from $\tilde{x}_1, \dots, \tilde{x}_{k-3}$.

If in \mathcal{G} there are vertices \tilde{x}^1, \tilde{x}^2 , adjacent to each of $\tilde{x}_1, \dots, \tilde{x}_{k-2}$, then the vertices \tilde{x}^1, \tilde{x}^2 are in one of the sets $V(\hat{X}_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$, $V(\hat{X}_{\mathcal{G}}^4(\tilde{x}_1, \dots, \tilde{x}_{k-3})) \setminus V(\hat{X}_{\mathcal{G}}^2(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$, $V(\hat{X}_{\mathcal{G}}^5(\tilde{x}_1, \dots, \tilde{x}_{k-3})) \setminus V(\hat{X}_{\mathcal{G}}^4(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$. Anyway without loss of generality we can consider the vertex \tilde{x}_{k-2} to be in $V(\hat{X}_{\mathcal{G}}^5(\tilde{x}_1, \dots, \tilde{x}_{k-3}))$. Spoiler chooses $\tilde{y}_{k-2} = \varphi(\tilde{x}_{k-2})$. Further choices of Duplicator are described in the same manner as for the case 1.

If there are no two vertices adjacent to each other and to each of $\tilde{x}_1, \dots, \tilde{x}_{k-2}$, then further reasonings are identical to reasonings from subcases of the case 1., in which we use safe pairs.

Lemma is proved.

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